Welcome to the archival Web page for U.C. Berkeley's **Physics 110B**, Section 1, Spring 2001. Email to: (Prof.) Mark Strovink, strovink@lbl.gov . I have a research web page, a standardized U.C. Berkeley web page, and a statement of research interests.

The problem set solutions for this course were composed by its Graduate Student Instructor Gesualdo Riday.

Course documents:
Most documents linked here are in PDF format and are intended to be displayed by Adobe Acrobat [Reader], version 4 or later (Acrobat will do a better job if you **un**check "Use Greek Text Below:" on File-Preferences-General).

General Information including schedules and rooms.
Course Outline.
Special Relativity Notes* used in Physics H7B/H7C, S/F '99.
* handwritten

Problem Set 1	Solution Set 1
Problem Set 2	Solution Set 2
Problem Set 3	Solution Set 3
Problem Set 4	Solution Set 4
Problem Set 5	Solution Set 5
Problem Set 6	Solution Set 6
Problem Set 7	Solution Set 7
Problem Set 8	Solution Set 8
Problem Set 9	Solution Set 9
Problem Set 10	Solution Set 10
Problem Set 11	Solution Set 11
Problem Set 12	Solution Set 12
Practice Midterm	
Midterm	Solution to Midterm
Practice Final	
Final Exam	Solution to Final Exam

Mark Strovink

Professor
Particle Experiment

Mark Strovink, Ph.D. 1970 (Princeton). Joined UC Berkeley faculty in 1973 (Professor since 1980). Elected Fellow of the American Physical Society; served as program advisor for Fermilab (chair), SLAC (chair), Brookhaven, and the U.S. Department of Energy; served as D-Zero Physics Coordinator (1997 & 1998).

Research Interests

I am interested in experiments using elementary particles to test discrete symmetries, absolute predictions and other fundamental tenets of the Standard Model. Completed examples include early measurement of the parameters describing charge parity (*CP*) nonconservation in *K* meson decay; establishment of upper limits on the quark charge radius and early observation of the effects of gluon radiation in deep inelastic muon scattering; and establishment of stringent limits on right-handed charged currents both in muon decay and in proton-antiproton collisions, the latter via the search for production of right-handed *W* bosons in the D-Zero experiment at Fermilab.

After the discovery in 1995 by CDF and D-Zero of the top quark, we measured its mass with a combined 3% error, yielding (with other inputs) a stringent test of loop corrections to the Standard Model and an early hint that the Higgs boson is light. If a Higgs-like signal is seen, we will need to measure the top quark mass more than an order of magnitude better in order to determine whether that signal arises from the SM Higgs.

Current Projects

A continuing objective is to understand better how to measure the top quark mass. Top quarks are produced mostly in pairs; each decays primarily to $b + W$. The b's appear as jets of hadrons. Each W decays to a pair of jets or to a lepton and neutrino. For top mass measurement the most important channels are those in which either one or both of the W's decay into an electron or muon. For the single-lepton final states, we developed in 1994-96 and applied in 1997 a new technique that suppresses backgrounds (mostly from single W production) without biasing the apparent top mass spectra. For the dilepton final states, where backgrounds and systematic errors are lower but two final-state neutrinos are undetected rather than one, a likelihood *vs.* top mass must be calculated for each event. During 1993-96 we developed a new prescription for this calculation that averages over the (unmeasured) neutrino rapidities, and we used it in 1997 to measure the top mass to ~7% accuracy in this more sparsely populated channel. In both channels, further improvements to measurement technique as well as accumulation of larger samples will be necessary.

While studying data from the 1992-1996 CDF and D-Zero samples that contain both an electron and a muon, we became aware of three events that cannot easily be attributed either to top quark decay or to backgrounds. Generally this is because the transverse momenta of the leptons (electrons, muons, and neutrinos as inferred from transverse momentum imbalance) are unexpectedly large. We anticipate confirming data *e.g.* from the D-Zero run that began in 2001.

Transverse momentum imbalance is a broad signature for new physics. For example, in many supersymmetric models, *R*-parity conservation requires every superparticle to decay eventually to a lightest superparticle that, like the neutrino, can be observed only by measuring a transverse momentum imbalance. Reliable detection of this signature is one of the severest challenges for collider detectors. D-Zero's uniform and highly segmented uranium/liquid argon calorimeter yields the best performance achieved so far. Building on that, we have developed a new approach to analysis of transverse momentum imbalance that, for a given efficiency, yields up to five times fewer false positives.

Recently we have grappled with the long-standing problem of searching with statistical rigor for new physics in samples that should be describable by Standard Model processes – when the signatures for new physics are *not* strictly predefined. We have identified plausible methods for performing this type of analysis, and have exercised them on D-Zero data, but the methods involve sacrifices in sensitivity that we are still working to mitigate.

Selected Publications

S. Abachi *et al*. (D-Zero Collaboration), "Search for right-handed W bosons and heavy W' in proton-antiproton collisions at $\sqrt{s}$ = 1.8 TeV," *Phys. Rev. Lett*. **76**, 3271 (1996).

S. Abachi *et al*. (D-Zero Collaboration), "Observation of the top quark," *Phys. Rev. Lett*. **74**, 2422 (1995).

B. Abbott *et al*. (D-Zero Collaboration), "Direct measurement of the top quark mass," *Phys. Rev. Lett*. **79**, 1197 (1997); *Phys. Rev.* **D 58**, 052001 (1998).

B. Abbott *et al*. (D-Zero Collaboration), "Measurement of the top quark mass using dilepton events," *Phys. Rev. Lett*. **80**, 2063 (1998); *Phys. Rev.* **D 60**, 052001 (1999).

V.M. Abazov *et al*. (D-Zero Collaboration), "A quasi-model-independent search for new high p_T physics at D-Zero," *Phys. Rev. Lett*. **86**, 3712 (2001); *Phys. Rev.* **D 62**, 092004 (2000); *Phys. Rev.* **D 64**, 012004 (2001).

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University of California, Berkeley
Physics 110B, Section 1, Spring 2001 (Strovink)

GENERAL INFORMATION (13 Feb 01)

Web site for this course: `http://d0lbln.lbl.gov/110bs01-web.htm` .

Instructors: Prof. **Mark Strovink**, 437 LeConte; (LBL) 486-7087; (home, before 10) 486-8079; (UC) 642-9685. Email: `strovink@lbl.gov` . Web: `http://d0lbln.lbl.gov` . Office hours: M 3:15-4:15, 5:30-6:30.
Mr. **Gesualdo Riday**, 279 LeConte, (UC) 642-5647. Email: `gesualdo_riday@yahoo.com` . Office hours (in 279 LeConte): W 3-4, Th 2-3.

Lectures: MWF 10:10-11:00 in 343 LeConte, and Tu 5:10-6:30 in 308 LeConte. The Tu 5:10-6:30 slot will be used occasionally during the semester for the midterm exam; for reviews and special lectures; and for lectures that substitute for those which would normally be delivered on F 10:10-11:00. Lecture attendance is strongly encouraged, since the course content is not exactly the same as that of the texts.

Discussion Sections: Tentatively M 1:10-2 in 331 LeConte, and W 5:10-6 in 5 Evans. Begin in second week. Taught by Mr. Riday. You are especially encouraged to attend discussion section regularly. There you will learn techniques of problem solving, with particular application to the assigned exercises.

Texts:
• Griffiths, **Introduction to Electrodyamics** (3rd ed., Prentice-Hall, 1999, required). Probably you already bought this book for 110A. If not, get the fourth (or later) printing, which has fewer typos. Most of you have already formed an opinion about this text, which I feel is well written and pedagogically effective, though its scope is modest and its problems are sometimes not very physical.
• Pedrotti & Pedrotti, **Introduction to Optics** (2nd ed., Prentice-Hall, 1993, required). There is no uniform choice of optics text for this course. Hecht, **Optics**; Fowles, **Introduction to Modern Optics**; and, for a heavy-duty treatment, Klein & Furtak, **Optics** all have been used in various incarnations of 110B.
• If you are planning to attend physics graduate school, it would be smart now to purchase Jackson, **Classical Electrodynamics** (3rd ed., Wiley). Optionally, it can be useful in this course.
• Optionally, Taylor & Wheeler, **Spacetime Physics** (Freeman, 1966, paperback) can be useful for the portion of this course that is devoted to special relativity.

Problem Sets: A required and most important part of the course. Twelve problem sets are assigned and graded. Problem sets are due on Thursdays at 5 PM, beginning in week 2. Deposit problem sets in the box labeled "110B Section 1 (Strovink)" in the second floor breezeway between LeConte and Birge Halls. You are encouraged to attempt all of the problems. Students who do not do so find it almost impossible to learn the material and to succeed on the examinations. Late papers will not be graded. Your lowest problem set score will be dropped, in lieu of due date extensions for any reason. You are encouraged to discuss problems with others in the course, but you must write up your homework by yourself. (In a small class it is straightforward to identify solutions that are written collectively; our policy is to divide the score among the collectivists.)

Exams: There will be one 80-minute midterm examination and one 3-hour final examination. Before confirming your enrollment in this class, please check that its final Exam Group 6 does not conflict with the Exam Group for any other class in which you are enrolled. Please verify now that you will be available for the midterm examination on Tu 20 Mar, 5:10-6:30 PM; and for the final examination on M 14 May, 8-11 AM. Except for unforeseeable emergencies, it will not be possible for the midterm or the final exam to be rescheduled. Passing 110B requires passing the final exam.

Grading: 25% problem sets, 25% midterm, 50% final exam. Departmental regulations call for an *A*:*B*:*C* distribution in the ratio 2:3:2, with approximately 10-15% *D*'s or *F*'s. However, the fraction of *D*'s or *F*'s depends on you; no minimum number need be given.

COURSE OUTLINE

Week No.	Week of..	Topic	Griffiths	Pedrotti	Problem Set No.	Due at 5 PM on..
1	15-Jan	MARTIN LUTHER KING HOLIDAY				
	17-Jan	FIRST LECTURE (review EM waves)	9.1-9.3			
		EM waves in conductors; mirrors	9.4			
2	22-Jan	Driven oscillator model for n(ω)	9.4			
		Waveguides	9.5			
		Lumped-element circuits			1	25-Jan
3	29-Jan	Alternating-current networks				
		Scalar and vector potentials	10.1			
		Lorentz and Coulomb gauge	10.1		2	1-Feb
4	5-Feb	Retarded potentials	10.2			
		Liénard-Wiechert potentials	10.3			
		Fields of a moving point charge	10.3		3	8-Feb
5	12-Feb	Special relativity	12			
		Special relativity	12			
		Special relativity	12		4	15-Feb
6	19-Feb	PRESIDENTS' HOLIDAY				
		Special relativity	12			
		Special relativity	12		5	22-Feb
7	26-Feb	Special relativity	12			
		Special relativity	12			
		Special relativity	12		6	1-Mar
8	5-Mar	Multipole radiation	11.1			
		Multipole radiation	11.1			
		Radiation by a point charge	11.2		7	8-Mar
9	12-Mar	Radiation by a point charge	11.2			
		Bremsstrahlung and synchrotron radiation	11.2			
		Cherenkov and transition radiation				
10	19-Mar	Matrix analysis of polarization		14		
	20-Mar	80-min **Midterm Exam**, Tu 5:10-6:30 PM				
		Matrix analysis of polarization		14		
		Interference and coherence		10,12		
	26-Mar	SPRING RECESS				
11	2-Apr	Interference and coherence		10,12		
		Interference and coherence		10,12		
		Multiple reflections		11,19	8	5-Apr
12	9-Apr	Multiple reflections		11,19		
		Fraunhofer diffraction		16		
		Fraunhofer diffraction		16	9	12-Apr
13	16-Apr	Diffraction grating		17		
		Fourier optics		25		
		Fourier optics		25	10	19-Apr
14	23-Apr	Fresnel diffraction		18		
		Holograms		13		
		Holograms		13	11	26-Apr
15	30-Apr	Lasers		21,22		
		Lasers		21,22		
		Lasers		21,22	12	3-May
16	7-May	LAST LECTURE (review)				
	11-May	Final examinations begin				
17	14-May	180-minute **Final Exam,** M 8-11 AM				
	19-May	Final examinations end				

Physics H7C

University of California, Berkeley

RELATIVITY NOTES

F'99 (Strovink)

1. SPECIAL RELATIVITY

1.1 SPACETIME

Because c = speed of light in vacuum is the same in all reference frames according to Maxwell's equations, we can imagine considering

$$ct = (m/sec)(sec) = (m)$$

to be the 0^{th} dimension in spacetime.

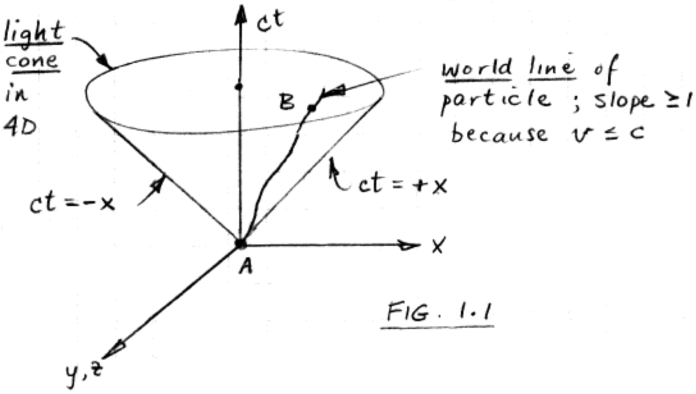


FIG. 1.1

An event is described by $r = (\overset{0}{ct}, \overset{1}{x}, \overset{2}{y}, \overset{3}{z})$. Because information travels at $\le c$, if event B is causally connected to event A, at the origin, event B must be within the light cone.

1.2 DISTANCE IN SPACETIME

What is r^2? Consider 2 non-accelerating ("inertial") reference frames

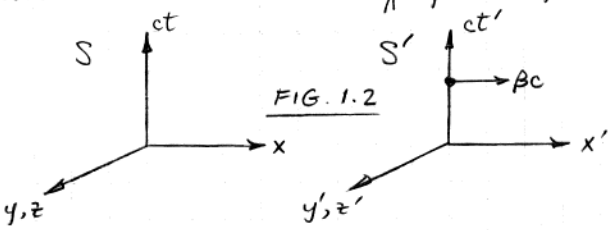


FIG. 1.2

We choose the origins to be the same, i.e. $x = y = z = 0$ is the same point as $x' = y' = z' = 0$ when $ct = ct' = 0$. Frame S′ is moving in the ($x = x'$) direction with respect to S with velocity βc.

A pulse of EM radiation is emitted at $(ct, x, y, z) = (ct', x', y', z') = 0$. In either frame it is a bubble expanding from the 3D origin:

$$x^2 + y^2 + z^2 = c^2t^2$$
$$x'^2 + y'^2 + z'^2 = c^2t'^2 .$$

$$\therefore\ c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 \qquad (1.1)$$

for the bubble.

So we define the distance Δr between 2 events r_A and r_B to be

$$\Delta r^2 = c^2(t_B - t_A)^2 \ominus (x_B - x_A)^2 \ominus (y_B - y_A)^2 \ominus (z_B - z_A)^2 \qquad (1.2)$$

Had we used $\oplus$ instead of $\ominus$, (1.1) would have forced the distance between 2 events to be different when viewed in different frames.

Distances between events are called
- timelike if $\Delta r^2 > 0$ $(c^2\Delta t^2 > |\Delta\vec{r}|^2)$
- lightlike " $=$ " $=$
- spacelike " $<$ " $<$.

Except for quantum mechanical effects, pairs of events can be causally connected only if the interval between them is timelike (within the light cone) or lightlike (on the light cone).

1.3 ROTATION IN 2D SPACE

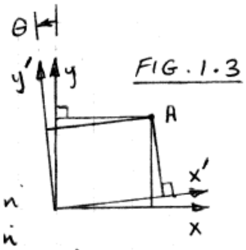


FIG. 1.3

$r = (x, y)$ and $r' = (x', y')$ are the coordinates of point A as viewed in S or S′. From the diagram, when $\theta \ll 1$ we obtain the infinitesimal transformation

$$x' = x + \theta y \qquad \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (1.3)$$
$$y' = -\theta x + y \qquad \text{or}$$

The distance between point A and the origin is

$$r^2 = x^2 + y^2$$
$$r'^2 = x'^2 + y'^2 = (x + \theta y)^2 + (y - \theta x)^2$$
$$= x^2 + \cancel{2\theta xy} + y^2 - \cancel{2\theta xy} + \mathcal{O}(\theta^2)$$
$$= r^2 \ \checkmark$$

neglect

When θ is not $\ll 1$, (1.3) becomes

$$\begin{aligned} x' &= \cos\theta\, x + \sin\theta\, y \\ y' &= -\sin\theta\, x + \cos\theta\, y \end{aligned} \quad \text{or} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad (1.4)$$

For non infinitesimal rotations it is still true that $r'^2 = r^2$ because $\sin^2\theta + \cos^2\theta = 1$.

1.4 INFINITESIMAL TRANSFORMATION IN 2D SPACETIME

$r = (ct, x)$ and $r' = (ct', x')$ are the coordinates of point A as viewed in S or S'. From the diagram, when $\beta \ll 1$,

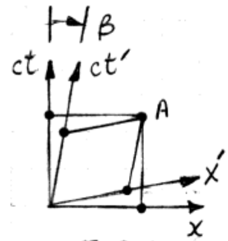


FIG 1.4

$$\begin{aligned} x' &= x \ominus \beta ct \\ ct' &= -\beta x + ct \end{aligned} \quad \text{or} \quad \begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \qquad (1.5)$$

Why did we draw the diagram in this peculiar way, requiring a $\ominus$ rather than the usual $\oplus$ sign where indicated? The distance between event A and the origin is

$$\begin{aligned} r^2 &= c^2t^2 - x^2 \\ r'^2 &= c^2t'^2 - x'^2 = (ct - \beta x)^2 - (x - \beta ct)^2 \\ &= (ct)^2 - \cancel{2\beta ctx} - x^2 \cancel{+ 2\beta ctx} \underbrace{+\, O(\beta^2)}_{\text{neglect}} \\ &= r^2 \ \checkmark \end{aligned}$$

The peculiar diagram is <u>necessary to force $r'^2 = r^2$</u>.

The nonrelativistic <u>Galilei transformation</u> is obtained from Eq. (1.5) by ignoring $-\beta x$ with respect to ct:

$$\begin{aligned} x' &= x - \beta ct = x - vt \\ t' &\approx t \end{aligned} \qquad (1.6)$$

You used this transformation (perhaps without realizing it) to solve distance = rate × time problems in high school.

The form of Eq. (1.5) which <u>exactly preserves distances in spacetime</u> is

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \frac{1}{\sqrt{1-\beta^2}} \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \qquad (1.7)$$

but this discussion so far pertains only to infinitesimal transformations.

1.5. FINITE TRANSFORMATION IN 2D SPACETIME

When β in Fig. 1.4 is not $\ll 1$, we call the rotation parameter η ("eta") rather than β. η is called the "rapidity" or "boost" and, in general, is a function of β.
When the spacetime transformation is no longer infinitesimal, Eq. (1.5) becomes

$$\begin{aligned} x' &= (\cosh\eta)x - (\sinh\eta)ct \\ ct' &= -(\sinh\eta)x + (\cosh\eta)ct \end{aligned}$$

or $\qquad (1.8)$

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$

using the <u>hyperbolic functions</u>

$$\cosh a \equiv \frac{e^{a} + e^{-a}}{2} \qquad \sinh a \equiv \frac{e^{a} - e^{-a}}{2}$$

$$\tanh a \equiv \sinh a / \cosh a$$

$$\sinh(0) = 0, \ \cosh(0) = 1, \ \tanh(0) = 0$$

$$\sinh(\infty) = \infty, \ \cosh(\infty) = \infty, \ \tanh(\infty) = 1$$

$$\cosh^2 a - \sinh^2 a = 1$$

It is the <u>last</u> property which guarantees $r'^2 = r^2$ for finite transformations in spacetime.

Rewrite Eq. (1.8) as

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \cosh\eta \begin{bmatrix} 1 & -\tanh\eta \\ -\tanh\eta & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \qquad (1.9)$$

and compare to Eq. (1.7) making use of

$$\cosh^2\eta = \frac{\cosh^2\eta}{\cosh^2\eta - \sinh^2\eta} = \frac{1}{1 - \tanh^2\eta}.$$

Eqs. (1.7) and (1.9) are consistent if
$$\beta = \tanh\eta \ (\le 1), \quad \eta = \tanh^{-1}\beta \qquad (1.10)$$
$\Rightarrow \exists$ <u>no</u> faster-than-light particles (tachyons).

Defining $\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}$,
Eq. (1.9) becomes the <u>Lorentz transformation</u>

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \quad \text{or} \qquad (1.11)$$

$$\begin{aligned} x' &= \gamma x - \gamma\beta ct \\ ct' &= -\gamma\beta x + \gamma ct. \end{aligned}$$

1.6. GENERALIZATIONS OF LORENTZ TRANSFORMATION

- 2D → 4D, $\vec{\beta}$ still along $\hat{x} = \hat{x}'$:

$$\underbrace{\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix}}_{r'} = \underbrace{\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{call this } \Lambda} \underbrace{\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}}_{r} \qquad (1.12)$$

or $r' = \Lambda r$.

- If $\vec{\beta}$ is along $\hat{n}$ rather than $\hat{x}$:

$$r' = \Lambda_R^{-1} \Lambda \Lambda_R r \qquad (1.13)$$

where Λ_R is a <u>3D spatial rotation</u> which transforms the $\hat{n}$ direction to the $\hat{x}$ direction :

$$\Lambda_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ 0 & \lambda_{yx} & \lambda_{yy} & \lambda_{yz} \\ 0 & \lambda_{zx} & \lambda_{zy} & \lambda_{zz} \end{bmatrix} \qquad (1.14)$$

[for a rotation $(\Lambda_R^{-1})_{ji} = (\Lambda_R)_{ij}$, i.e. Λ_R is <u>orthogonal</u>].

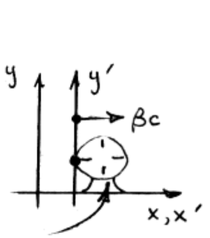
- If $\vec{\beta}$ is along $-\hat{x}$ instead of x, change the sign of β in (1.12). That is,

if $r' = \Lambda(\beta) r$ ← <u>direct</u> L.T.
(Λ is a function of β)

then $r = \Lambda(-\beta) r'$.
↑ <u>inverse</u> Lorentz transformation.

1.7. TIME DILATION

As usual, S and S′ have the same spatial origin at $t = t' = 0$



clock at rest in S′

$$\Delta t' \equiv \underset{\substack{\uparrow \\ 2^{nd}\text{ ring}}}{t_2'} - \underset{\substack{\uparrow \\ 1^{st}\text{ ring}}}{t_1'}$$

as observed at <u>fixed x'</u>.

Using <u>inverse</u> Lorentz transformation,

$$ct_2 = \gamma ct_2' + \gamma\beta x_2' \qquad \text{but } x_2' = x_1'$$
$$ct_1 = \gamma ct_1' + \gamma\beta x_1' \qquad \text{(clock fixed in S}')$$

$$c\Delta t = \gamma c \Delta t', \quad \boxed{\Delta t = \gamma \Delta t'} \qquad (1.15)$$

∴ Since γ always ≥ 1 the interval between rings is always <u>longer</u> when measured in a frame which is moving with respect to S′, where the two events occur at the <u>same place</u>.

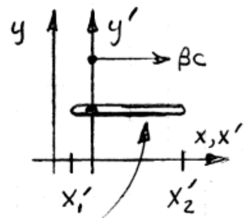
Since S′ is a <u>unique</u> frame in which to observe the time interval between these two events, we assign a unique name to this time interval :

$\Delta t' \equiv \Delta\tau \equiv$ "<u>proper time</u>" interval

$$\Delta t = \gamma \Delta\tau$$
$$dt = \gamma d\tau \qquad \text{"time dilation"} \qquad (1.16)$$

(Note that the observer in S uses his/her <u>own</u> fine grid of clocks and data loggers to measure Δt.)

1.8. SPACE CONTRACTION



rod at rest in S′

$\Delta x' \equiv x_2' - x_1'$, measured at any t'.

The observer in S, with his/her <u>own</u> fine grid of clocks, rulers, and data loggers, measures the positions x_1 and x_2 of the two ends of the rod at the <u>same time</u> $t_1 = t_2$.

Using <u>direct</u> L.T.,

$$x_2' = \gamma x_2 - \gamma\beta ct_2 \qquad \text{but } t_2 = t_1$$
$$x_1' = \gamma x_1 - \gamma\beta ct_1$$

$$\Delta x' = \gamma \Delta x \qquad \boxed{\Delta x = \Delta x'/\gamma} \qquad (1.17)$$

The length of the rod as observed in a system moving with respect to it is always <u>shorter</u> than its proper length $\Delta x'$.

The analysis of {1.7} could have been done with the direct L.T., and that of {1.8} with the inverse L.T. More algebra would be required to get the same result.

1.9. EINSTEIN LAW OF VELOCITY ADDITION

The identity

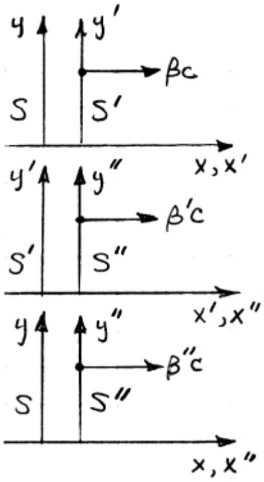
$$\tanh(a+b) = \frac{\tanh a + \tanh b}{1+\tanh a \tanh b} \qquad (1.18)$$

can be believed in analogy to the well known

$$\tan(a+b) = \frac{\tan a + \tan b}{1 \ominus \tan a \tan b}$$

or it can be derived in a few lines using

$$\tanh a \equiv \frac{e^{a}-e^{-a}}{e^{a}+e^{-a}} \qquad (1.19)$$



Problem: along the $\hat{x} = \hat{x}' = \hat{x}''$ direction,

S'	has velocity	βc	wrt S
S''		$\beta' c$	S'
S''		$\beta'' c$	S

Given β and β', what is β''?

Since the boost parameter η is additive, we know

$$\eta'' = \eta + \eta'$$

But $\beta = \tanh \eta$ etc.

$$\therefore \quad \beta'' = \tanh\eta'' = \tanh(\eta+\eta')$$

$$= \frac{\tanh\eta + \tanh\eta'}{1+\tanh\eta\tanh\eta'} \quad \text{using (1.18)}$$

$$\boxed{\beta'' = \frac{\beta+\beta'}{1+\beta\beta'}} \qquad \text{(Einstein law)} \qquad (1.20)$$

1.10 SPACE TRAVEL: HUMAN CONSTRAINTS

Assume that an astronaut
- must be accelerated at $\leq 1g$
- must age ≤ 40 years during the voyage.

What maximum velocity can be achieved? How far can he/she travel, and how much time will have elapsed on earth?

The voyage consists of 10 years with $a_x' = +g$, 20 years $a_x' = -g$, 10 years $a_x' = +g$. We consider only the first leg. To get the answers to the questions above, we then double the first leg distance and quadruple the first leg time.

Suppose that the astronaut at a certain moment has an $\hat{x}$ velocity equal to βc. The astronaut is accelerating and so his/her rest frame is not inertial. To analyze his/her motion using the Lorentz transformation we need an inertial frame, so we define a comoving frame S' which is instantaneously at rest with respect to the astronaut but which is not accelerating.

In an infinitesimal proper time interval $d\tau$ (= same in astronaut and comoving frames, since relative $\gamma_{rel} = 1$ to 2nd order in β_{rel}), the astronaut's velocity increases, relative to comoving frame, by $g\,d\tau$:

$$g\,d\tau = dv_{rel} \equiv c\,d\beta_{rel}$$

Since $\beta_{rel} = 0$ and $d\beta_{rel}$ is infinitesimally small,

$$d\eta_{rel} \cong d\beta_{rel}$$

where η is the boost parameter. Since the boost parameter is additive, as seen on the earth (frame S)

$$\eta(\tau + d\tau) = \eta(\tau) + d\eta_{rel}$$

$$d\eta/d\tau \cong d\beta_{rel}/d\tau = g/c$$

$$\eta_{max} = \int_0^{\tau_0 = 10\,yrs} \frac{d\eta}{d\tau}\,d\tau$$

$$= \int_0^{\tau_0} \frac{g}{c}\,d\tau = \frac{g\tau_0}{c} = 10.34^{*}$$

$$\beta_{max} = \tanh\eta_{max} = 1 - (2.09 \times 10^{-9}).$$

* The most boosted particles in accelerators (electrons at LEP) have $\eta \approx 12.2$.

The distance covered is

$$dx = \beta c\, dt = (\tanh\eta) c (\gamma d\tau) \quad \text{using time dilation}$$
$$= c(\tanh\eta)(\cosh\eta)\, d\tau$$
$$= c \sinh\eta\, d\tau$$
$$\Delta x = \underbrace{2}_{\text{2 legs}} c \int_0^{\tau_0} \sinh\eta\, d\tau = 2c\int_0^{\tau_0} \sinh\left(\frac{g\tau}{c}\right) d\tau$$
$$= 2\frac{c^2}{g}\left(\cosh\frac{g\tau_0}{c} - 1\right) \quad \text{meters}$$
$$= 2.84\times 10^{20} \text{ meters}$$

$= 29{,}900$ light years, or $\approx 2\times 10^{-7}$ the size of the universe. So only $\approx 10^{-20}$ of it can be explored by man.

The time elapsed on earth is

$$dt = \gamma d\tau = \cosh\eta\, d\tau$$
$$\Delta t = \underbrace{4}_{\text{4 legs}} \int_0^{\tau_0} \cosh\frac{g\tau}{c}\, d\tau = 4\frac{c}{g}\sinh\frac{g\tau_0}{c}$$
$$= 1.89\times 10^{12} \text{ sec}$$
$$= 59{,}850 \text{ yrs} \quad (\text{compare } 40 \text{ yrs.})$$

This last result is called the "twin paradox." It is not a paradox because the earthbound twin is not accelerating.

1.11 FOUR-MOMENTUM

If we wish to write Eq. (1.2) in the form

$$(r_B - r_A)^2 = c^2(t_B-t_A)^2 - (x_B-x_A)^2 - (y_B-y_A)^2 - (z_B-z_A)^2$$
$$\equiv r_B{}^2 - 2r_B\cdot r_A + r_A{}^2$$

with $r_B \cdot r_A \equiv$ inner product of 2 4-vectors in spacetime, it must be the case that

$$r_B\cdot r_A \equiv c^2 t_B t_A - x_B x_A - y_B y_A - z_B z_A$$

and that the inner product of any 2 4-vectors is independent of reference frame (invariant to Lorentz transformations).

The proper time interval $d\tau$ and the rest mass m are also Lorentz invariants.

Form $p \equiv m\dfrac{dr}{d\tau}$, that is

$$(p_0, p_x, p_y, p_z) = \left(mc\frac{dt}{d\tau}, m\frac{dx}{d\tau}, m\frac{dy}{d\tau}, m\frac{dz}{d\tau}\right).$$

Note that $dt/d\tau = \gamma$ so $dx/d\tau = \gamma\, dx/dt$. So

$$p = (\gamma mc, \gamma m v_x, \gamma m v_y, \gamma m v_z) \qquad (1.21)$$
$$\equiv (E/c, \vec{p})$$

must transform like r, i.e. must also be a 4-vector. It is called the four-momentum. We can write

$$\begin{bmatrix} E'/c \\ p_x' \\ p_y' \\ p_z' \end{bmatrix} = \Lambda \begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix} \quad \text{with } \Lambda \text{ as in } (1.12).$$

The length^2 of p is Lorentz invariant and we can evaluate it in a frame in which the CM of the system it describes is not moving ($\vec{p} = 0, \gamma = 1$). Then

$$p^2 = \underbrace{E^2/c^2 - |\vec{p}|^2}_{\text{true in any frame}} = \underbrace{m^2c^2}_{\text{rest frame value}} \qquad (1.22)$$

This is the fundamental equation for solving relativistic kinematics problems.

What is E? Make a Taylor series expansion

$$E = \frac{mc^2}{(1-v^2/c^2)^{1/2}} = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right)$$

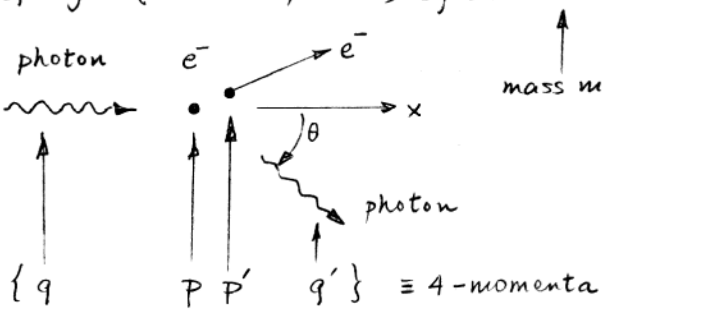
For $v \ll c$ this is $E = mc^2 + \frac{1}{2}mv^2$ where the last term is the nonrelativistic kinetic energy. We interpret the first term as the rest mass energy:

$$\underbrace{E}_{\text{total energy}} = \gamma mc^2 = \underbrace{mc^2}_{\text{rest mass energy}} + \underbrace{T}_{\text{kinetic energy}} \qquad (1.23)$$

We see the possibility of converting mass to energy (lots of energy because c^2 is large).

1.12 COMPTON (PHOTON–ELECTRON) SCATTERING

To illustrate the power of Eq.(1.22) for solving problems in relativistic kinematics, we consider the scattering of a quantum of light (massless photon) by an electron at rest.



$p = (mc, \vec{0})$ because target electron at rest.

$q = (q_0, q_x, 0, 0)$

Since photon massless, $q \cdot q = 0$ by Eq. (1.22), so $q_x = q_0$ and we can write

$q = (q_0, q_0, 0, 0)$; $\quad q' = (q_0', q_0'\cos\theta, q_0'\sin\theta, 0)$

Problem: what is the relationship between the final photon energy cq_0' and its final angle θ wrt $\hat{x}$?

Use energy-momentum conservation $\Rightarrow$ 4-momentum conservation:

$$q + p = q' + p' \quad \text{(this is 4 equations!)}$$

$$q - q' + p = p'$$

$$[(q-q') + p]\cdot[(q-q') + p] = p' \cdot p'$$

$$(q-q')\cdot(q-q') + 2p\cdot(q-q') + p\cdot p = p'\cdot p'$$

$$\underset{0}{\underset{\parallel}{q\cdot q}} - 2q\cdot q' + \underset{0}{\underset{\parallel}{q'\cdot q'}} + 2p\cdot(q-q') + \underset{\cancel{(mc)^2}}{\underset{\parallel}{p\cdot p}} = \underset{\cancel{(mc)^2}}{\underset{\parallel}{p'\cdot p'}}$$

$$\cancel{2}p\cdot(q-q') = \cancel{2}q\cdot q'$$

$$(mc, 0, 0, 0)\cdot(q_0 - q_0', \; q_0 - q_0'\cos\theta, \; -q_0'\sin\theta, 0) =$$

$$= (q_0, q_0, 0, 0)\cdot(q_0', q_0'\cos\theta, q_0'\sin\theta, 0)$$

$$mc(q_0 - q_0') = q_0 q_0'(1 - \cos\theta) \qquad \div\, q_0 q_0' mc:$$

$$\boxed{\frac{1}{q_0'} - \frac{1}{q_0} = \frac{1}{mc}(1 - \cos\theta)} \qquad (1.24)$$

This is A.H. Compton's famous formula. Conventionally it is multiplied by Planck's constant h, with the photon wavelength $\lambda = h/q_0$. Then

$$\boxed{\lambda' - \lambda = \lambda_c(1 - \cos\theta)} \text{ with} \qquad (1.25)$$

$$\lambda_c \equiv h/mc \, .$$

λ_c, the Compton wavelength of the electron, is

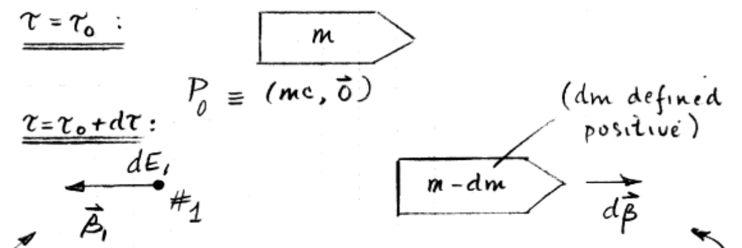
$$\lambda_c = 2\pi \times 386 \times 10^{-15}\, m$$

Planck's constant is

$$h = 2\pi \times 6.58 \times 10^{-16} \text{ eV sec.}$$

1.13 SPACE TRAVEL: PROPULSION CONSTRAINTS

Again consider the spacecraft as viewed in the comoving frame (§1.10). In infinitesimal proper time interval $d\tau$ the rocket motor ejects particle #1 with energy dE_1 and relative velocity $\beta_1 c$.



$$p_1 \equiv \left(\frac{dE_1}{c}, \vec{\beta}_1 \frac{dE_1}{c}\right) \qquad \left((m-dm)c + \frac{mc|d\vec{\beta}|^2}{2}, \; mc\,d\vec{\beta}\right) \equiv P'$$

In assigning the 4-momentum to particle #1, we used $p = (\gamma mc, \gamma\vec{\beta}mc)$ so that $\vec{p}/p_0 = \vec{\beta}$. In assigning the 4-momentum to the spacecraft, we used the fact that (as viewed in the comoving frame) the spacecraft is nonrelativistic, so that $E \approx mc^2 + \frac{1}{2}mv^2$.

If we assume a perfectly efficient engine, i.e. no heat energy radiated in random directions, both energy and momentum will be conserved:

$$P_0 = p_1 + P' \qquad \text{neglect, 2nd order in infinitesimals.}$$

$$\cancel{mc} = \frac{dE_1}{c} + (\cancel{m} - dm)c + \cancel{\frac{mc}{2}|d\vec{\beta}|^2} \qquad \left(\begin{matrix}\text{timelike}\\\text{part}\end{matrix}\right)$$

$$\vec{0} = \vec{\beta}_1 \frac{dE_1}{c} + mc\,d\vec{\beta} \qquad \left(\begin{matrix}\text{spacelike}\\\text{part}\end{matrix}\right)$$

Substituting $\frac{dE_1}{c} = c\,dm$ from the timelike eq$^{\text{n}}$, the spacelike eq$^{\text{n}}$ becomes

$$|d\vec{\beta}| = |\vec{\beta}_1| \frac{dm}{m} \, .$$

Again we set $|d\vec{\beta}| \approx d\eta$, where η is the boost, since rocket is nonrelativistic in comoving frame.

As additional particles (#2, #3, etc) are ejected, the boosts $d\eta_i$ are <u>additive</u>.

$$\therefore \quad \eta_{final} - (\eta_0 \equiv 0) = \int_{m_0}^{m_{final}} \beta_1 \frac{|dm|}{m}$$

$$\boxed{\eta_{final} = \beta_1 \ln \frac{m_0}{m_{final}}} \qquad (1.26)$$

<u>Chemical rocket engines</u> achieve maximum $\beta_1 \approx 4 \times 10^3 \text{ m/sec}/c \approx 1.33 \times 10^{-5}$. Then to achieve a boost of 10.34 (see §1.10) requires

$$\ln \frac{m_0}{m_f} = 7.8 \times 10^5$$

$$m_f = m_0 \times (\text{a number beyond calculator range}).$$

Chemical engines <u>will not suffice</u>.

<u>Relativistic engines</u> emit particles at $\beta_1 \cong 1$. If they were unit efficient,

$$\ln \frac{m_0}{m_f} = 10.34$$

$$m_f = 3.1 \times 10^4 \, m_0 .$$

<u>Manned payload</u> requires $m_f \geq 10T$ for life support; then

$$m_0 \geq 3.1 \times 10^5 \, T$$

$\Rightarrow$ a rocket heavier than an aircraft carrier. ($\leq 10^5 T$)

Note that Eq. (1.26) becomes

$$\eta_{final} = \underline{\epsilon} \beta_1 \ln \frac{m_0}{m_{final}} \qquad (1.27)$$

if the efficiency ϵ of the engine is not unity.

Present relativistic engine concepts...

- are grossly inefficient ($\epsilon \ll 1$)
- leave most of their fuel on board so that m_f/m_0 cannot be $\ll 1$.

(Example: laser powered by batteries)

1.14 OTHER FOUR-VECTORS

In addition to

$$r = (ct, \vec{r})$$

$$p = (E/c, \vec{p}) \qquad (E \equiv \gamma mc^2, \ \vec{p} \equiv \gamma m\vec{v}),$$

frequently encountered other 4-vectors are

$$\partial \equiv \left(\frac{\partial}{c\partial t}, -\vec{\nabla}\right) \qquad \left(\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)$$

$$k \equiv \left(\frac{\omega}{c}, \vec{k}\right) \quad \text{as in } e^{i(\omega t - \vec{k}\cdot\vec{r})} \qquad (1.28)$$

"wave vector"

ω = angular freq.

$$A \equiv (\varphi, \vec{A}) \qquad (1.29)$$

"vector potential"

$$\text{where} \begin{cases} \vec{B} \equiv \vec{\nabla} \times \vec{A} \\ \vec{E} \equiv -\vec{\nabla}\varphi - \dfrac{\partial \vec{A}}{c\partial t} \end{cases}.$$

Because dot products of 4-vectors are Lorentz invariant, so are

$$k \cdot r \equiv \omega t - \vec{k}\cdot\vec{r} \quad \text{"phase of a wave"}$$

$$\partial \cdot \partial \equiv \Box = \frac{\partial^2}{c^2 \partial t^2} - \nabla^2 \quad \text{"D'Alembertian"}$$

$$\partial \cdot A \equiv \frac{\partial \phi}{c\partial t} + \vec{\nabla}\cdot\vec{A} \quad (= 0 \text{ when } \vec{A} \text{ satisfies the "Lorentz gauge condition"})$$

When the "de Broglie momentum" equation $|\vec{p}| = h/\lambda$ is combined with the "Planck frequency" equation $E = h\nu$, using

$$|\vec{k}| \equiv 2\pi/\lambda , \qquad (1.30)$$

both equations can be summarized by

$$(E/c, \vec{p}) \equiv \boxed{p = \frac{h}{2\pi} k} \equiv \frac{h}{2\pi}\left(\frac{\omega}{c}, \vec{k}\right) \qquad (1.31).$$

(this is 4 equations)

"generalized de Broglie eq<u>n</u>"

Another 4-vector is

$$j \equiv (c\rho, \vec{J}) \qquad \begin{matrix} \rho = \text{chg density (esu/cm}^3) \\ \vec{J} = \text{current density (esu/cm}^2\text{-sec)} \end{matrix}$$

$$\text{Lorentz invariant} \Big\} \ \partial \cdot j = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \qquad (1.32)$$

(charge conservation)

1.15 LORENTZ TRANSFORMATION OF ELECTROMAGNETIC FIELDS

The fact that

$$\begin{bmatrix}\phi' \\ A_x' \\ A_y' \\ A_z'\end{bmatrix} = \begin{bmatrix}\gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\phi \\ A_x \\ A_y \\ A_z\end{bmatrix}$$

and

$$\left.\begin{aligned}\vec{B} &= \vec{\nabla}\times\vec{A} \\ \vec{E} &= -\vec{\nabla}\phi - \frac{\partial \vec{A}}{c\,\partial t}\end{aligned}\right\}\text{ cgs units!}$$

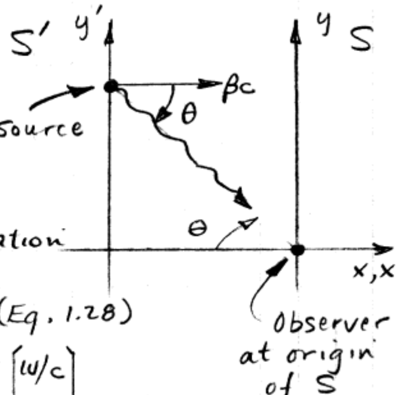
leads after some algebra to the following transformation equations for $\vec{E}$ and $\vec{B}$:

$$\begin{aligned}\vec{E}_\perp' &= \gamma(\vec{E}_\perp + \vec{\beta}\times\vec{B}) \\ \vec{B}_\perp' &= \gamma(\vec{B}_\perp - \vec{\beta}\times\vec{E}) \\ \vec{E}_\parallel' &= \vec{E}_\parallel\ ,\quad \vec{B}_\parallel' = \vec{B}_\parallel\end{aligned} \qquad (1.33)$$

where "$\perp$" means $\perp$ to $\hat{\beta}$ and "$\parallel$" means parallel to $\hat{\beta}$.

A consequence of Eq. (1.33) is that $|\vec{E}|^2 - |\vec{B}|^2$ is a Lorentz invariant.

1.16 RELATIVISTIC DOPPLER SHIFT



Apply the direct Lorentz transformation (Eq. 1.12) to the wave 4-vector k (Eq. 1.28)

$$\begin{bmatrix}\omega'/c \\ k_x' \\ k_y' \\ k_z'\end{bmatrix} = \Lambda\begin{bmatrix}\omega/c \\ k_x \\ k_y \\ k_z\end{bmatrix}$$

$$\Rightarrow \frac{\omega'}{c} = \gamma\frac{\omega}{c} - \gamma\beta k_x \qquad (1.34)$$

Let the lab phase velocity of the wave be $\beta_s c$ ($\beta_s = 1$ for a light wave). ($T \equiv$ period)

$$|\vec{k}| = \frac{2\pi}{\lambda} = \frac{2\pi}{\beta_s c T} = \frac{2\pi\nu}{\beta_s c} = \frac{\omega}{\beta_s c}$$

so in Eq. (1.34) we may write

$$k_x = |\vec{k}|\cos\theta = \frac{\omega}{\beta_s c}\cos\theta .\quad \text{Then}$$

$$\omega' = \gamma\omega\left(1 - \frac{\beta}{\beta_s}\cos\theta\right)$$

$$\boxed{\omega = \frac{\omega'}{\gamma\left(1 - \frac{\beta}{\beta_s}\cos\theta\right)}} \quad \text{Relativistic Doppler shift} \qquad (1.35)$$

Special cases:

- light wave $\Rightarrow \beta_s = 1$

$$\boxed{\omega = \frac{\omega'}{\gamma(1-\beta\cos\theta)}} \qquad (1.36)$$

- $\left.\begin{matrix}\text{approaching} & (\theta = 0) \\ \text{receding} & (\theta = \pi)\end{matrix}\right\}$ light wave:

$$\omega = \frac{\omega'}{\gamma(1\mp\beta)} = \left(\frac{1\pm\beta}{1\mp\beta}\right)^{1/2}\omega'$$

- $\cos\theta = 0$ (source is at zenith, where nonrelativistically there is no Doppler shift):

$$\omega = \frac{\omega'}{\gamma}\ ,\quad T = T'\gamma \quad \left(\begin{matrix}\text{ordinary} \\ \text{time} \\ \text{dilation}\end{matrix}\right)$$

- $\beta \ll 1$

$$\omega \cong \frac{\omega'}{1 - \frac{\beta}{\beta_s}\cos\theta} = \frac{\omega'}{1 - \frac{v_{source}}{v_{wave}}\cos\theta}$$

(= freshman physics Doppler shift. Note sonic boom at $\cos\theta = v_{wave}/v_{source}$.)

University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

Problem Set 1

1. Griffiths 9.11.

2. Griffiths 9.18.

3. Griffiths 9.19.

4. Griffiths 9.20.

5. At the rate of 1 card/sec, psychic Uri Geller (`http://skepdic.com/geller.html`) turns over each card in a deck. He communicates by "paranormal" means the identity of each card to his assistant, from whom he is shielded with respect to sound and visible light.

As a physicist, you consider all EM waves to be normal. To test the notion that Uri's talents defy the laws of physics, you resolve to design a shield that will prevent Uri from using any relevant EM frequency to communicate with his assistant.
(**a**) Roughly what minimum EM frequency must Uri use? (*Hint*: Consider that a 56 kbps modem operates over audio telephone frequencies.)
(**b**) Design a spherical shell, enclosing a volume of 1 m^3 for Uri's comfort, that will attenuate the EM waves generated by Uri's brain to $\approx 1/400 \approx e^{-6}$ of their original amplitude. Use the minimum EM frequency that you calculated in (a).
(**c**) How much does your shield weigh? (Try to design the lightest shield that will do the job. Does it help to use a ferromagnetic material?)

6. An electromagnetic cavity can be considered to be just another resonant oscillator, with a quality factor Q equal to the ratio of the energy stored to the energy dissipated during the time interval $\Delta t = 1/\omega_0$. Consider a cubical box of side d whose inner surfaces are plated with an adequate thickness of silver, which is an excellent conductor. This cavity has a fundamental resonant angular frequency equal to

$$\omega_0 = \frac{c}{d} \times \pi\sqrt{2} \ ,$$

where the first factor can be identified from purely dimensional arguments, and the second factor, a function of the cavity's geometry, is of order unity. Apart from a different geometrical factor of order unity, the Q of this cavity turns out to be of order

$$Q \approx \frac{V}{A\kappa^{-1}} \ ,$$

where V is the cavity's volume, A is its inside surface area, and κ^{-1} is the skin depth. Thus, Q is of the same order as the ratio of the cavity's volume to its "skin depth volume".
(**a**) Taking $d = 10$ cm, what Q can be achieved?
(**b**) If the cavity is kept at the same size, would it help to operate it at one of its higher frequency modes?
(**c**) If the cavity is always operated at its fundamental frequency, would it help to build it bigger?

7. Show that the results in Griffiths Eq. (9.147) are equivalent to the familiar formulæ

$$\begin{aligned} R &= \frac{Z_2 - Z_1}{Z_2 + Z_1} \\ T &= \frac{2Z_2}{Z_2 + Z_1} \ , \text{ where} \\ Z &\equiv \frac{\tilde{E}_0}{\tilde{H}_0} \ , \\ R &\equiv \frac{\tilde{E}_{0_R}}{\tilde{E}_{0_I}} \ , \text{ and} \\ T &\equiv \frac{\tilde{E}_{0_T}}{\tilde{E}_{0_I}} \ , \end{aligned}$$

and where Z is the characteristic impedance of the medium, R is the amplitude reflection coefficient, and T is the amplitude transmission coefficient.

8. Consider a dilute material with $\epsilon = \epsilon_0$ and $\mu = \mu_0$, but with slight conductivity $\sigma = \beta\epsilon_0\omega$, where $\beta \ll 1$ is a constant. EM radiation of angular frequency ω is normally incident from vacuum upon this material.
(**a**) Relative to the incident field, show that the reflected electric field has a magnitude of $\beta/4$ and a phase shift of 90°.
(**b**) Show that the transmitted wave is attenuated with a skin depth equal to $\lambda_0/2\pi$ divided by β, where λ_0 is the vacuum wavelength, and that its **H** lags **E** by a phase shift equal to $\beta/2$.

Physics 110B
Homework Solution #1

#1. (Griffiths 9.11)

$$\langle fg \rangle = \frac{1}{T}\int_0^T A\cos(k\cdot r-\omega t+\delta_a)\,B\cos(k\cdot r-\omega t+\delta_b)\,dt$$

$$= \frac{AB}{2T}\int_0^T dt\left(\cos(k\cdot r-\omega t+\delta_a+k\cdot r-\omega t+\delta_b)+\cos(k\cdot r-\omega t+\delta_a-k\cdot r+\omega t-\delta_b)\right)$$

$$= \frac{AB}{2T}\int_0^T dt\left(\cos(2k\cdot r-2\omega t+\delta_a+\delta_b)+\cos(\delta_a-\delta_b)\right)$$

$$= \frac{AB}{2\cancel{T}}\cos(\delta_a-\delta_b)\cancel{T} = \frac{AB}{2}\cos(\delta_a-\delta_b)$$

Now, in complex notation:

$$\tilde{f} = A\exp(i(k\cdot r-\omega t+\delta_a)), \quad \tilde{g} = B\exp[i(k\cdot r-\omega t+\delta_b)]$$

$$\frac{1}{2}\tilde{f}\tilde{g}^* = \frac{1}{2}AB^*\exp[i(k\cdot r-\omega t+\delta_a-k\cdot r+\omega t-\delta_b)]$$

$$= \frac{1}{2}AB^*\exp[i(\delta_a-\delta_b)]$$

Thus,

$$\boxed{\mathrm{Re}\left(\tfrac{1}{2}\tilde{f}\tilde{g}^*\right) = \tfrac{1}{2}AB\cos(\delta_a-\delta_b) = \langle fg \rangle}$$

#2. (Griffiths 9.18)

a) $\rho_f(t) = \exp(-\sigma/\epsilon)t\, \rho_f(0)$ (eq 9.120)

$\tau_{\text{time to flow to the surface}} = \dfrac{\epsilon}{\sigma}$ $\epsilon = \epsilon_0 \epsilon_r$, $\epsilon_r \approx n^2$ (eq 9.70)

$\approx \epsilon_0 n^2 / \sigma \approx \epsilon_0 n^2 / (1/\rho)$ $n_{glass} = 1.5$ average index of refraction for glass

$\approx 8.85\times 10^{-12} \frac{C^2}{Nm^2} (1.5)^2 / (10^{-12}\ \Omega m)$ ← from (Table 7.1)

$\boxed{\tau = 20 \text{ seconds}}$

b) For silver: $\rho = 1.59\times 10^{-8}$ (Table 7.1) $\epsilon \approx \epsilon_0$ $\omega = 10^{10} Hz \cdot 2\pi$

$\sigma = 1/\rho = 6.25\times 10^{7} \gg \omega\epsilon = 2\pi\times 10^{10} \times 8.85\times 10^{-12} = .56$

Thus, eq (9.126) $K = \omega\sqrt{\frac{\epsilon\mu}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right)^{1/2} \approx \sqrt{\frac{\omega\sigma\mu}{2}}$

And thus, the skin depth is:

eq(9.128) $d = \frac{1}{K} \cong \sqrt{\frac{2}{\omega\sigma\mu}} = \sqrt{\frac{2}{2\pi\times 10^{10} \times 6.25\times 10^{7} \times 4\pi\times 10^{-7}}} = 6.4\times 10^{-4}$ mm

Therefore the silver coating should be $\boxed{1.0\times 10^{-6} m}$

c) $\sigma_{cu} = 1/1.68\times 10^{-8} = 6\times 10^{7}$ (Table 7.1) ; $\omega\epsilon_0 = (2\pi\times 10^{6})\cdot(8.85\times 10^{-12}) = 6\times 10^{-5}$

$\sigma \gg \omega\epsilon$ therefore, $K \approx \sqrt{\frac{\omega\sigma\mu}{2}}$ eq (9.126)

And,

$\lambda_{\text{in copper}} = 2\pi\sqrt{\frac{2}{\omega\sigma\mu_0}} = 2\pi\sqrt{\frac{2}{2\pi\times 10^{6} \times 6\times 10^{7} \times 4\pi\times 10^{-7}}} = \boxed{4\times 10^{-4} M}$ (eq 9.129)

$V_{\text{in copper}} = \lambda\nu = 4\times 10^{-4} \times 10^{6} = \boxed{400 M/s}$

in vacuum

$\lambda = c/\nu = \frac{3\times 10^{8}}{10^{6}} = \boxed{300 M}$

$V = c = \boxed{3\times 10^{8} M/s}$

#3. (Griffiths 9.19)

a) $\kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}$ (eq 9.126)

when $\omega\epsilon \gg \sigma$ we use the binomial expansion for the square root:

$$\approx \omega\sqrt{\frac{\epsilon\mu}{2}}\left(1+\frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2-1\right)^{1/2}$$

$$= \cancel{\omega}\sqrt{\frac{\epsilon\mu}{2}}\,\frac{1}{\sqrt{2}}\,\frac{\sigma}{\epsilon\cancel{\omega}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$$

$$d = \boxed{\frac{1}{\kappa} \cong \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}}$$ (eq 9.128)

For pure water, $\begin{cases} \epsilon = \epsilon_r\epsilon_0 = 80.1\epsilon_0 & \text{(Table 4.2)} \\ \mu = \mu_0(1+\chi_m) = \mu_0(1-9.0\times10^{-6}) \approx \mu_0 & \text{(Table 6.1)} \\ \sigma = 1/2.5\times10^5 & \text{(Table 7.1)} \end{cases}$

So, $d \approx \frac{2}{1/2.5\times10^5}\sqrt{\frac{80.1\,(8.85\times10^{-12})}{4\pi\times10^{-7}}} = \boxed{1.19\times10^4\,m}$

b) When $\sigma \gg \omega\epsilon$

$\kappa \approx k \approx \omega\sqrt{\frac{\cancel{\epsilon}\mu}{2}}\sqrt{\frac{\sigma}{\cancel{\epsilon}\omega}} = \sqrt{\frac{\omega\mu\sigma}{2}}$ (eq 9.126)

$\lambda = 2\pi/k \approx 2\pi/\kappa = 2\pi d$, or $\boxed{d = \frac{\lambda}{2\pi}}$

$d = \frac{1}{\kappa} \approx \sqrt{\frac{2}{\omega\mu\sigma}}$ $\quad \omega \approx 10^{15} \quad \epsilon \approx \epsilon_0 \quad \mu = \mu_0 \quad \sigma \approx 10^7\,(\Omega m)^{-1}$

$$= \sqrt{\frac{2}{10^{15}\cdot 4\pi\times10^{-7}\cdot 10^7}} = 1/8\times10^7 = \boxed{1.3\times10^{-8}\,m}$$

• Therefore, light does not penetrate far into the metal - which accounts for its opacity.

c) from part (b) we had $k \approx \kappa$

Also, $\varphi \equiv \tan^{-1}(\kappa / k)$ (eq 9.134)

Therefore $\boxed{\varphi = \tan^{-1}(1) = 45^\circ}$

$$\frac{B_0}{E_0} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}} \approx \boxed{\sqrt{\frac{\sigma\mu}{\omega}}} \qquad \sigma \gg \omega\epsilon \qquad \text{eq}(9.137)$$

For a typical metal: $\dfrac{B_0}{E_0} = \sqrt{\dfrac{10^7 \cdot 4\pi \times 10^{-7}}{10^{15}}} = \boxed{10^{-7}\ \text{s/m}}$

#4. (Griffiths 9.20)

a) $\langle u \rangle = \frac{1}{2}\langle \epsilon E^2 + \frac{1}{\mu}B^2 \rangle$

$$= \frac{1}{2}e^{-2\kappa z}\left\langle \epsilon E_0^2 \cos^2(kz-\omega t+\delta_E) + \frac{1}{\mu}B_0^2\cos^2(kz-\omega t+\delta_E+\varphi)\right\rangle$$

eq (9.138)

$$= \frac{1}{2}e^{-2\kappa z}\left(\frac{\epsilon}{2}E_0^2 + \frac{1}{2\mu}B_0^2\right)$$

$$= \frac{1}{4}e^{-2\kappa z}\left(\epsilon E_0^2 + \frac{1}{\mu}E_0^2\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\right) \quad \text{eq } (9.137)$$

$$= \frac{1}{4}e^{-2\kappa z}\epsilon E_0^2\left(1+\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\right)$$

$$= \frac{1}{4}e^{-2\kappa z}\epsilon E_0^2\left(\frac{k}{\omega}\sqrt{\frac{2}{\epsilon\mu}}\right)^2 \quad \text{(eq 9.126)}$$

$$= \frac{1}{4}e^{-2\kappa z}\epsilon E_0^2\left(\frac{k^2}{\omega^2}\frac{2}{\epsilon\mu}\right)$$

$$\boxed{\langle u \rangle = \frac{k^2}{2\mu\omega^2}E_0^2 e^{-2\kappa z}}$$

$$\frac{\langle U_{magnetic} \rangle}{\langle U_{electric} \rangle} = \frac{B_0^2/\mu}{E_0^2 \epsilon} = \frac{1}{\mu\epsilon} \mu\epsilon \sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} \quad \text{(eq 9.137)}$$

$$\boxed{\frac{\langle U_{mag} \rangle}{\langle U_{elec} \rangle} = \sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} > 1}$$

magnetic contribution always dominates

b)

$$I = \langle S \rangle = \langle \frac{1}{\mu} E \times B \rangle = \frac{1}{\mu} \langle E_0 B_0 e^{-2\kappa z} \cos(kz - \omega t + \delta_E) \cos(kz - \omega t + \delta_E + \phi)\, \hat{z} \rangle$$

$$= \frac{1}{2\mu} E_0 B_0 e^{-2\kappa z} \cos\phi \quad \text{(From problem \#1)} \qquad \text{(eq 9.138)}$$

$$= \frac{1}{2\mu} E_0^2 e^{-2\kappa z} \left(\frac{K}{\omega} \cos\phi\right) \quad \text{(eq 9.135)}$$

Also, $\frac{\kappa}{k} = \tan\phi \implies \cos\phi = \frac{k}{(k^2+\kappa^2)^{1/2}} = \frac{k}{K}$

Thus, $\boxed{I = \frac{k}{2\mu} E_0^2 e^{-2\kappa z}}$

#5

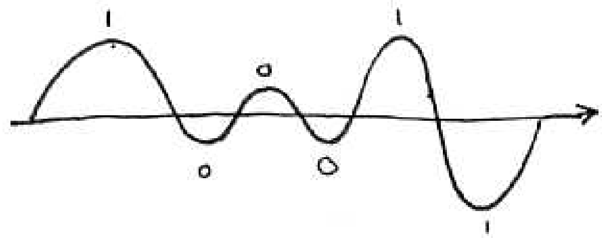
a) In a deck there are 52 cards, to indicate which card he has picked he will need 6 bits.

Since, $52 \approx 2^6 = 64$

Thus, Uri needs to send data at the speed of,

$$\frac{1 \text{ card}}{\text{second}} \cdot \frac{6 \text{ bits}}{\text{card}} = \boxed{6 \text{ bits}/\text{second}}$$

We will assume that the information is sent in a digital fashion (high/low peak) via an EM wave,



$\Leftarrow$ this is a signal transmitting 110001 (6 $\frac{\text{bits}}{\text{second}}$ of information)

The above signal has a frequency $\boxed{\nu = 3\,Hz}$

This roughly the minimum frequency Uri can use, however, to be sure about the accuracy of the data he should send a wave w/ twice the data capacity of 6Hz.

b) The skin depth equals:

$$\delta = \frac{1}{\kappa} = \frac{1}{\omega}\sqrt{\frac{2}{\epsilon\mu}}\left(\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right)^{-1/2} \approx \frac{1}{\omega}\sqrt{\frac{2}{\epsilon\mu}}\sqrt{\frac{\epsilon\omega}{\sigma}} \quad \sigma \gg \epsilon\omega$$

$$= \sqrt{\frac{2}{\mu\sigma\omega}}$$

To attenuate the EM waves generated by Uri's brain to $\approx \frac{1}{400} = e^{-6}$

we will need a shell with a width of six skin depths:

$\boxed{\text{width of shell} = 6\delta}$

We have the following physical constants for Al, Cu, and Fe:

	σ	μ	$\rho_{density}$ g/cm³
Al	3.77×10^7	μ_0	2.699
Cu	5.95×10^7	μ_0	8.96
Fe	1.04×10^7	$500\mu_0$	7.874

$\mu_0 = 4\pi\times10^{-7}$

$\omega = 18$

Using the above values

$\boxed{Al: 6\delta = .28m}$, $\boxed{Cu: 6\delta = .23m}$, $\boxed{Fe: 6\delta = .024m}$

In order to allow V_{r_i} to have $1\,m^3$ of volume the shell has to have a radius equal to:

$$\frac{4}{3}\pi r_1^3 = 1\,m^3 \Rightarrow \boxed{r_1 = .62\,m}$$

The mass of the different metal shells equals:

$$mass = \frac{4}{3}\pi\left((r_1 + 6\delta)^3 - r_1^3\right)\rho_{density}$$

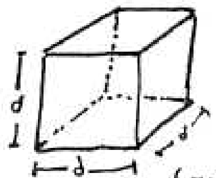
$$\boxed{\begin{array}{l} Al: \ mass = 5{,}500\ kg \\ Cu: \ mass = 14{,}000\ kg \\ Fe: \ mass = 950\ kg \end{array}}$$

Thus, we see that the ferromagnetic iron shell is the lightest.

$\boxed{\text{Use Iron}}$

#6

a)



$$\omega_0 = \frac{c}{d} \times \pi\sqrt{2} = \frac{3\times10^8}{.1} \times \pi\sqrt{2} = 1.3\times10^{10}$$

(o>>ε) good conductor
• we get this result from problem #2

$$K \cong \sqrt{\frac{\omega_0 \mu \sigma}{2}} = \sqrt{\frac{1.3\times10^{10} \cdot 4\pi\times10^{-7} \cdot 6.3\times10^7}{2}} = \underline{\underline{7.3\times10^5\,m^{-1}}}$$

Now, $Q \cong \frac{V}{AK^{-1}} = \frac{d^3}{6d^2 K^{-1}} = \frac{d}{6K^{-1}}$

V : cavity's volume $- d^3$
A : cavity's inside surface area $- 6d^2$

$$\boxed{Q \approx \frac{.1 \cdot 7.3\times10^5}{6} = 1.2\times10^4}$$

b) $Q \approx \frac{dK}{6} = \frac{d}{6}\sqrt{\frac{\omega\mu\sigma}{2}} \Rightarrow$ Thus, we see that if ω goes up so will Q

Therefore, $\boxed{\text{increase } \omega \text{ to increase } Q}$

c) $Q \approx \frac{d}{6}\sqrt{\frac{\omega\mu\sigma}{2}} \Rightarrow$ Therefore, $\boxed{\text{yes it helps to increase } Q \text{ if } d \text{ is increased}}$

#7

$$R \equiv \frac{\tilde{E}_{oR}}{\tilde{E}_{oI}} = \frac{1-\tilde{\beta}}{1+\tilde{\beta}} \quad (\text{eq } 9.147)$$

$$\tilde{\beta} \equiv \frac{\mu_1 V_1}{\mu_2 V_2} = \frac{\mu_o V_1}{\mu_2 \omega / \tilde{k}_2} \quad \begin{array}{c}(\text{eq } 9.106)\\ \text{and}\\ (\text{eq } 9.146)\end{array}$$

$$\boxed{\tilde{\beta} = \frac{Z_1}{Z_2}}$$

$$Z_1 = \frac{\tilde{E}_{oI}}{\tilde{H}_{oI}} = \frac{\tilde{E}_{oI} e^{i(kz-\omega t)}}{\frac{1}{\mu_o V_1} \tilde{E}_{oI} e^{i(kz-\omega t)}} \quad \text{eq}(9.140)$$

$$Z_1 = \mu_o V_1$$

$$Z_2 = \frac{\tilde{E}_{oT}}{\tilde{H}_{oT}} = \frac{\tilde{E}_{oT} e^{i(kz-\omega t)}}{\frac{1}{\mu_2}\frac{\tilde{k}_2}{\omega} \tilde{E}_{oT} e^{i(kz-\omega t)}} \quad \text{eq}(9.141)$$

$$Z_2 = \mu_2 \omega / \tilde{k}_2$$

Thus,

$$R \equiv \frac{1-\tilde{\beta}}{1+\tilde{\beta}} = \frac{1 - Z_1/Z_2}{1+Z_1/Z_2} = \boxed{\frac{Z_2 - Z_1}{Z_2 + Z_1} = R}$$

$$T \equiv \frac{\tilde{E}_{oT}}{\tilde{E}_{oI}} = \frac{2}{1+\tilde{\beta}} \quad \text{eq}(9.147)$$

$$= \frac{2}{1+Z_1/Z_2} = \boxed{\frac{2Z_2}{Z_2+Z_1} = T}$$

#8

a) Given Information:

- $\mu_2 = \mu_o$, $\epsilon_2 = \epsilon_o$, $\sigma = \beta \epsilon_o \omega$ where $\beta \ll 1$
- EM radiation from vacuum is normally incident upon this material

$$\frac{E_{oR}}{E_{oI}} = \frac{1-\tilde{\beta}}{1+\tilde{\beta}} \qquad \tilde{\beta} = \frac{\mu_1 V_1}{\mu_2 \omega} \tilde{k}_2 = \frac{\mu_1 V_1}{\mu_2 \omega}(k_2 + i\kappa_2) \quad (\text{eq } 9.146)$$

$$k_2 \equiv \omega \sqrt{\frac{\epsilon_o \mu_o}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\epsilon \omega}\right)^2} + 1 \right)^{1/2} = \omega \sqrt{\frac{\epsilon_o \mu_o}{2}} \left(\sqrt{1 + \underbrace{\beta^2}_{0,\ \beta \ll 1}} + 1 \right)^{1/2} \quad (\text{eq } 9.126)$$

$$\cong \omega \sqrt{\frac{\epsilon_o \mu_o}{2}} (2)^{1/2} = \omega \sqrt{\epsilon_o \mu_o}$$

$$K_2 = \omega\sqrt{\frac{\epsilon_2\mu_2}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right)^{1/2} \approx \omega\sqrt{\frac{\epsilon_0\mu_0}{2}}\left(1+\tfrac{1}{2}\beta^2-1\right)^{1/2}$$

$$= \omega\sqrt{\epsilon_0\mu_0}\ \beta/2$$

Therefore,

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2\omega}(k_2+iK_2) = \frac{\cancel{\mu_0} v_1}{\cancel{\mu_0}\cancel{\omega}}\cancel{\omega}\sqrt{\epsilon_0\mu_0}\,(1+i\beta/2) = \frac{1}{\sqrt{\epsilon_0\mu_0}}\sqrt{\epsilon_0\mu_0}\,(1+i\beta/2)$$

$$\boxed{\tilde{\beta} = (1+i\beta/2)}$$

$$\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{1-\tilde{\beta}}{1+\tilde{\beta}} = \frac{1-1-i\beta/2}{1+1+i\beta/2} = -i\,\frac{\beta/2}{2+i\beta/2}$$

$$\approx -i\beta/4\,(1-i\beta/4) = -i\beta/4 - \beta^2/16$$

$$\boxed{\tilde{E}_{0R}/\tilde{E}_{0I} \approx -i\beta/4}$$

the negative i indicates a phase lag of 90° of the reflected wave relative to the incident wave.

b) From part (a)

$$K = \omega\sqrt{\epsilon_0\mu_0}\,\beta/2 = \frac{\omega}{c}\frac{\beta}{2} = \frac{2\pi\nu}{c}\frac{\beta}{2} = \frac{2\pi}{\lambda_0}\frac{\beta}{2}$$

So, $\boxed{\text{skin depth} = K^{-1} = \frac{\lambda_0}{\pi}\frac{1}{\beta}}$

eq (9.134) $\varphi_{Lag} = \tan^{-1}\left(\frac{K_2}{k_2}\right) = \tan^{-1}\left(\frac{\cancel{\omega\sqrt{\epsilon_0\mu_0}}\ \beta/2}{\cancel{\omega\sqrt{\epsilon_0\mu_0}}}\right)$

$= \tan^{-1}(\beta/2)$ $\quad\quad \beta \ll 1$

$$\boxed{\varphi_{lag} \approx \beta/2}$$

Therefore, H lags E by a phase shift equal to $\beta/2$.

University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

Problem Set 2

1. Griffiths 9.21.

2. Griffiths 9.22.

3. Griffiths 9.23.

4. Griffiths 9.24.

5. Griffiths 9.38. Consider the TE modes only.

6. Griffiths 9.31 part (b) only.

7. Show that the characteristic impedance $Z_0 \equiv \Delta V/I$ of the coaxial cable in the previous problem is

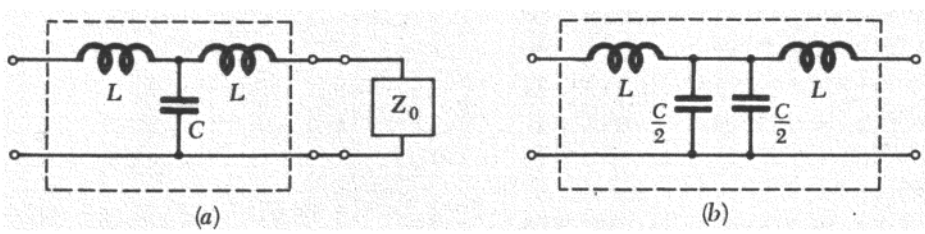
$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\frac{b}{a}$$

and that this result is equivalent to

$$Z_0 = \sqrt{\frac{L'}{C'}}$$

where L' and C' are the cable's inductance and capacitance per unit length, respectively.

8. In the circuit (a), an impedance Z_0 is to be connected to the terminals on the right.
(**a**) For given frequency ω, find the value that Z_0 must have if the resulting impedance between the left terminals is also to be Z_0. You should find that the required Z_0 is a (frequency-dependent) *pure resistance* R provided that $\omega^2 < 2/LC$.



(**b**) A chain of such boxes can be connected together to form a so-called ladder network. If the chain is terminated with a resistor of the correct (frequency-dependent) value R, show that its input impedance at frequency $\omega < \sqrt{2/LC}$ will continue to be R, regardless of the number of boxes that are added to the chain. (This type of ladder circuit is called a *lumped-element delay line*. In the low-frequency limit, the delay line's characteristic impedance reduces to

$$Z_0 \to \sqrt{\frac{L'}{C'}}\ ,$$

where $L' \propto 2L$ is the inductance per unit length, and $C' \propto C$ is the capacitance per unit length.)
(**c**) What is Z_0 in the special case $\omega = \sqrt{2/LC}$? (You may find it helpful to note that the contents of the box (a) are equivalent to those of the box (b).)

Physics 110 B Homework #2

#1 (Griffiths 9.21)

$$R = \left|\frac{\tilde{E}_{oR}}{\tilde{E}_{oI}}\right|^2 = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}^*}{1+\tilde{\beta}^*}\right) \quad \text{(eq 9.147)}$$

$$\tilde{\beta} \equiv \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2 \quad \text{(eq 9.146)}$$

$$= \frac{\mu_1 v_1}{\mu_2 \omega}(k_2 + i\kappa_2) \quad \text{(eq 9.125)}$$

$$k \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right)^{1/2}, \quad \kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right)^{1/2}$$

(eq 9.126)

Since, $\sigma \gg \epsilon\omega$ because silver is a good conductor

$$k_2 \approx \kappa_2 \equiv \omega\sqrt{\frac{\epsilon_2\mu_2}{2}}\sqrt{\frac{\sigma}{\epsilon_2\omega}} = \sqrt{\frac{\sigma\omega\mu_2}{2}}$$

So, $$\beta = \frac{\mu_1 v_1}{\mu_2 \omega}\sqrt{\frac{\sigma\omega\mu_2}{2}}(1+i) = \mu_1 v_1\sqrt{\frac{\sigma}{2\mu_2\omega}}(1+i)$$

$$A \equiv \mu_1 v\sqrt{\frac{\sigma}{2\mu_2\omega}} = \mu_0 c\sqrt{\frac{\sigma}{2\mu_0\omega}} = c\sqrt{\frac{\sigma\mu_0}{2\omega}} = 3\times10^8\sqrt{\frac{6\times10^7\cdot 4\pi\times10^{-7}}{2(4\times10^{15})}}$$

$$= 29$$

Therefore,

$$R = \left(\frac{1-\gamma-i\gamma}{1+\gamma+i\gamma}\right)\left(\frac{1-\gamma+i\gamma}{1+\gamma-i\gamma}\right) = \frac{(1-\gamma)^2+\gamma^2}{(1+\gamma)^2+\gamma^2} = \boxed{.93}$$

93% percent of the light is reflected.

#2 (Griffiths 9.22)

a) $v = \omega/k$. we are told $v = \alpha\sqrt{\lambda}$ where α is a constant

thus, $\frac{\omega}{k} = \alpha\sqrt{\frac{2\pi}{k}} \implies \omega = \alpha\sqrt{2\pi k}$

So, $v_{group} = \frac{d\omega}{dk} = \frac{\alpha}{2}\sqrt{\frac{2\pi}{k}} = \frac{\alpha}{2}\sqrt{\lambda} = v/2$

Or, $\boxed{v_{group} = v/2}$

b) $\Psi(x,t) = A\exp i(px - Et)/\hbar = A\exp i(kx - \omega t)$

$\Rightarrow \hbar k = p$, $E = \hbar\omega = p^2/2M$

Therefore, $v = \frac{\omega}{k} = \frac{E/\hbar}{p/\hbar} = \frac{p^2/2M}{p} = \frac{p}{2M} = \boxed{\frac{\hbar k}{2M} \text{ wave velocity}}$

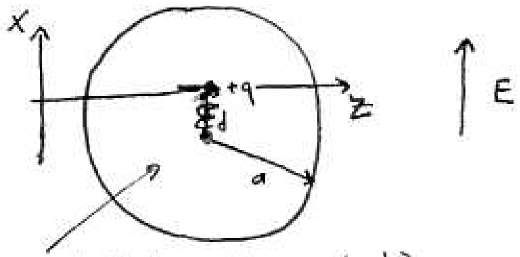
Also, $v_{group} = \frac{d\omega}{dk} = \frac{d}{dk}\left(\frac{\hbar k^2}{2M}\right) = \boxed{\frac{\hbar k}{M} \text{ group velocity}}$

$= \frac{p}{M}$ or $p = M v_{group} = M v_{classical}$

Thus, $\boxed{v_{group} = v_{classical}}$, and $\boxed{v = v_{group}/2}$

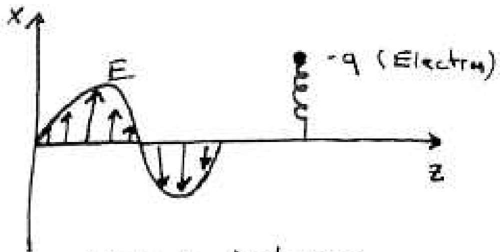
#3 (Griffiths 9.23)

From model in Ex 4.1



-q (electron charge cloud) distance varies by d.

This is analogous to the model in (figure 9.21)



electrons distance varies by x.

Therefore, from example 4.1

$$E = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3} \Rightarrow F = -qE = -\frac{1}{4\pi\epsilon_0}\frac{q^2 x}{a^3} = -k_{spring}\,x = -M\omega_0^2 x$$

(eq 9.151)

So, $\boxed{\omega_0 = \sqrt{\frac{q^2}{4\pi\epsilon_0 M a^3}}}$

$$\nu_0 = \omega_0/2\pi = \frac{1}{2\pi}\sqrt{\frac{(1.6\times10^{-19})^2}{4\pi(8.85\times10^{-12})(9.11\times10^{-31})(.5\times10^{-10})^3}} = \boxed{7.16\times10^{15}\ Hz}$$

$$\lambda = c/\nu_0 = 3\times10^8/7\times10^{15} = 3/7\times10^{-7} = 50nm$$

this is in the $\boxed{\text{ultraviolet}}$ region of the spectrum

Now,

$$n = 1 + \frac{Nq^2}{2M\epsilon_0}\frac{f_0}{\omega_0^2}\left(1 + \frac{\omega^2}{\omega_0^2}\right) \quad \text{(eq 9.173)}$$

$$n = 1 + A\left(1 + \frac{B}{\lambda^2}\right) \quad \text{(eq 9.174)}$$

So,

$$A = \frac{Nq^2}{2M\epsilon_0}\frac{f_0}{\omega_0^2},$$

N = # of molecules per unit vol. $= \frac{6.022\times10^{23}}{22.4L} = 6.022\times10^{23}/22.4\times10^{-3}M^3 = 2.69\times10^{25}$

f = # of electrons at the natural frequency ω_0 $= 2$

$$= \frac{2.69\times10^{25}(1.6\times10^{-19})^2}{(9.11\times10^{-31})(8.85\times10^{-12})(4.5\times10^{16})^2}$$

$$= \boxed{4.2\times10^{-5}}$$ (which is about 1/3 the actual value)

$$B = \frac{\omega^2\lambda^2}{\omega_0^2} = \left(\frac{2\pi c}{\omega_0}\right)^2 = \left(\frac{2\pi\times3\times10^8}{4.5\times10^{16}}\right)^2 = \boxed{1.8\times10^{-15}\ M^2}$$

(which is about 1/4 the actual value)

#4 (Griffiths 9.24)

In (Figure 9.22) the width of the anomalous dispersion region goes from ω_1 to ω_2 at these two points $n-1$ is at a maximum and a minimum respectively.

Thus, $n \cong 1 + \dfrac{Nq^2}{2m\epsilon_0} \dfrac{(\omega_0^2-\omega^2)}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}$ the denominator we label A

Since, ω_1 and ω_2 are maxs and mins we set the derivative of the index of refraction equal to zero

$$\frac{dn}{d\omega} = \frac{Nq^2}{2m\epsilon_0}\left\{\frac{-2\omega}{A} - \frac{\omega_0^2-\omega^2}{A^2}\left(2(\omega_0^2-\omega^2)(-2\omega)+\gamma^2 2\omega\right)\right\} = 0$$

$$2\omega D = (\omega_0^2-\omega^2)\left(2(\omega_0^2-\omega^2)-\gamma^2\right)2\omega$$

$$(\omega_0^2-\omega^2)^2+\gamma^2\omega^2 = 2(\omega_0^2-\omega^2)^2-\gamma^2(\omega_0^2-\omega^2)$$

$$(\omega_0^2-\omega^2)^2 = \gamma^2(\omega^2+\omega_0^2-\omega^2) = \gamma^2\omega_0^2$$

$$\omega_0^2-\omega^2 = \pm\omega_0\gamma \qquad \gamma \ll \omega_0$$

$$\omega = \sqrt{\omega_0^2 \mp \omega_0\gamma} = \omega_0\sqrt{1+\gamma/\omega_0} \approx \omega_0(1+\gamma/2\omega_0)$$

Therefore, $\omega_1 = \omega_0 - \gamma/2$, $\omega_2 = \omega_0 + \gamma/2$

And, $\boxed{\Delta\omega = \omega_2 - \omega_1 = \gamma}$

$$\alpha = 2\kappa \approx \frac{Nq^2\omega^2}{m\epsilon_0 c}\frac{\gamma}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} \qquad \text{(eq 9.171)}$$

when $(\omega=\omega_0)$ $\alpha_{max} = \dfrac{Nq^2}{m\epsilon_0 c\gamma}$ at ω_1 and ω_2, $\omega^2 = \omega_0^2 \mp \omega_0\gamma$ respectively

So, $\alpha = \dfrac{Nq^2\omega^2}{m\epsilon_0 c}\dfrac{\gamma}{\gamma^2\omega_0^2+\gamma^2\omega^2} = \alpha_{max}\left(\dfrac{\omega^2}{\omega^2+\omega_0^2}\right)$

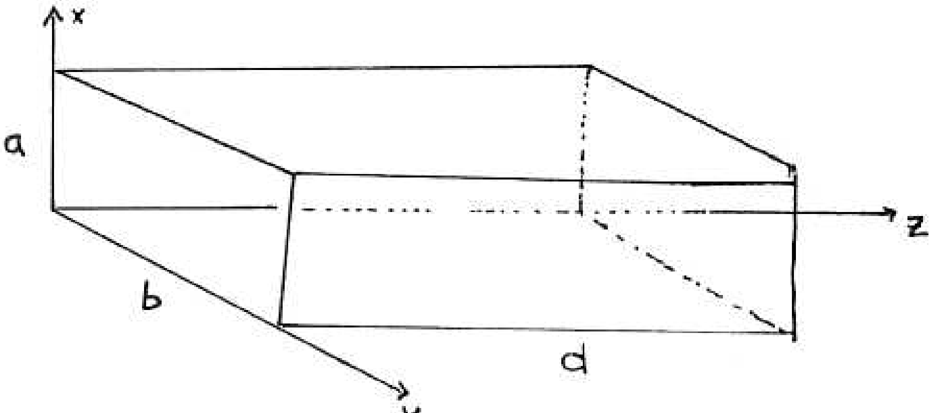
$\gamma \ll \omega_0$

Also, $\dfrac{\omega^2}{\omega^2+\omega_0^2} = \dfrac{\omega_0^2 \mp \omega_0\gamma}{2\omega_0^2 \mp \omega_0\gamma} = \dfrac{1}{2}\dfrac{(1 \mp \gamma/\omega_0)}{(1\mp \gamma/2\omega_0)} \approx \dfrac{1}{2}\left(1 \mp \dfrac{\gamma}{\omega_0}\right)\left(1 \pm \dfrac{\gamma}{2\omega_0}\right)$

$= \dfrac{1}{2}\left(1 \pm \dfrac{\gamma}{2\omega_0} \mp \dfrac{\gamma}{\omega_0} - \dfrac{\gamma^2}{2\omega_0^2}\right) \approx \dfrac{1}{2}\left(1 \mp \dfrac{\gamma}{2\omega_0}\right) \approx \dfrac{1}{2}$

Therefore, $\boxed{\alpha \approx \frac{1}{2}\alpha_{Max} \text{ at } \omega_1 \text{ and } \omega_2}$

5 (Griffiths 9.38)



Since, we have a resonant cavity we will have standing waves, therefore (eq 9.176) now will read:

i) $\tilde{E}(x,y,z,t) = \tilde{E}_0(x,y)\exp(-i\omega t)$

ii) $\tilde{B}(x,y,z,t) = \tilde{B}_0(x,y)\exp(-i\omega t)$

Thus, we note to get the correct equations for our standing waves we replace $-ik \Rightarrow \dfrac{\partial}{\partial z}$ in the equations on page 406-407

We thus have:

$$\left.\begin{aligned} &\text{i)}\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{c}\right)^2\right) E_z = 0 \\ &\text{ii)}\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{c}\right)^2\right) B_z = 0 \end{aligned}\right\} \text{(eq 9.181)}$$

Looking at transverse magnetic waves $B_z = 0$ we solve the above equation by seperation of variables:

$$E_z(x,y,z) = X_1(x)\, Y_1(y)\, Z_1(z)$$

So, $$X_1 Y_1 \frac{d^2 Z_1}{dz^2} + Z_1 Y_1 \frac{d^2 X_1}{dx^2} + X_1 Z_1 \frac{d^2 Y_1}{dy^2} + \left(\frac{\omega}{c}\right)^2 = 0$$

i) $\frac{1}{X_1}\frac{d^2 X_1}{dx^2} = -k_x^2$, ii) $\frac{1}{Y_1}\frac{d^2 Y_1}{dy^2} = -k_y^2$, iii) $\frac{1}{Z_1}\frac{d^2 Z_1}{dz^2} = -k_z^2$

The general solutions to (i), (ii), and (iii) are:

$$X_1(x) = A_1 \sin(k_x x) + B_1 \cos(k_x x)$$

$$Y_1(y) = C_1 \sin(k_y y) + D_1 \cos(k_y y)$$

$$Z_1(z) = E_1 \sin(k_z z) + F_1 \cos(k_z z)$$

By the symmetry of the problem we will get the following similar results for E_x and E_y:

$$E_y(x,y,z) = X_2(x)\, Y_2(y)\, Z_2(z)$$

$$\left.\begin{aligned} X_2(x) &= A_2 \sin(k_x x) + B_2 \cos(k_x x) \\ Y_2(y) &= C_2 \sin(k_y y) + D_2 \cos(k_y y) \\ Z_2(z) &= E_2 \sin(k_z z) + F_2 \cos(k_z z) \end{aligned}\right\} \text{ For } E_y(x,y,z)$$

$$E_x(x,y,z) = X_3(x)\, Y_3(y)\, Z_3(z)$$

$$\left.\begin{aligned} X_3(x) &= A_3 \sin(k_x x) + B_3 \cos(k_x x) \\ Y_3(y) &= C_3 \sin(k_y y) + D_3 \cos(k_y y) \\ Z_3(z) &= E_3 \sin(k_z z) + F_3 \cos(k_z z) \end{aligned}\right\} \text{ for } E_x(x,y,z)$$

Now, the above results need to meet the boundary condition $E_{\parallel} = 0$

Thus, when $z=0$ and $z=d$; $E_z \neq 0$, $E_y = 0$, $E_x = 0$

therefore, $F_2 = F_3 = 0$ $\quad E_1 = 0$ $\quad$ with $k_z = \ell\pi/d$

when $y=0$ and $y=b$; $E_y \neq 0$, $E_z = 0$, $E_x = 0$

So, $D_1 = D_3 = C_2 = 0$ $\quad$ with $k_y = n\pi/b$

Similarly for $x=0$ and $=a$; $E_x \neq 0$, $E_y = 0$, $E_z = 0$

$$B_1 = B_2 = A_3 = 0 \quad \text{with } k_x = m\pi/a$$

Putting all these results together we have,

$$\boxed{\begin{aligned} \bar{E} &= B\cos(k_x x)\sin(k_y y)\sin(k_z z)\,\hat{x} \\ &+ D\sin(k_x x)\cos(k_y y)\sin(k_z z)\,\hat{y} \\ &+ F\sin(k_x x)\sin(k_y y)\cos(k_z z)\,\hat{z} \end{aligned} \qquad \begin{aligned} &\text{with} \\ &k_x = m\pi/a \\ &k_y = n\pi/b \\ &k_z = \ell\pi/d \end{aligned}}$$

The magnetic field we get from $\bar{B} = -i/\omega\, \bar{\nabla} \times \bar{E}$

Therefore,
$$B_x = -i/\omega \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right)$$
$$B_y = -i/\omega \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right)$$
$$B_z = -i/\omega \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

Plugging the electric field from above we get

$$\boxed{\begin{aligned} \bar{B} = &-i/\omega\,(Fk_y - Dk_z)\sin(k_x x)\cos(k_y y)\cos(k_z z)\,\hat{x} \\ &-i/\omega\,(Bk_z - Fk_x)\cos(k_x x)\sin(k_y y)\cos(k_z z)\,\hat{y} \\ &-i/\omega\,(Dk_x - Bk_y)\cos(k_x x)\cos(k_y y)\sin(k_z z)\,\hat{z} \end{aligned}}$$

Also, from the altered eq 9.181 at the beginning of this problem we have:

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = c^2\left((m\pi/a)^2 + (n\pi/b)^2 + (\ell\pi/d)^2\right)$$

Or,

$$\boxed{\omega = c\pi\sqrt{(m/a)^2 + (n/b)^2 + (\ell/d)^2}}$$

#6 (Griffiths 9.31)

part (b) only

For a coaxial transmission line:

$$\left.\begin{aligned} E(s,\varphi,z,t) &= \frac{A\cos(kz-\omega t)}{s}\,\hat{s} \\ B(s,\varphi,z,t) &= \frac{A\cos(kz-\omega t)}{cs}\,\hat{\varphi} \end{aligned}\right\}(\text{eq } 9.197)$$

To determine $\lambda(z,t)$ we use Gauss's law

$$\oint E\cdot da = \frac{\cos(kz-\omega t)}{s}\,2\pi s\cdot z = \frac{1}{\epsilon_0}Q_{encl} = \frac{1}{\epsilon_0}\lambda z$$

$$\Rightarrow \boxed{\lambda(z,t) = 2\pi\epsilon_0 A\cos(kz-\omega t)}$$

To determine $I(z,t)$ we use Ampere's Law

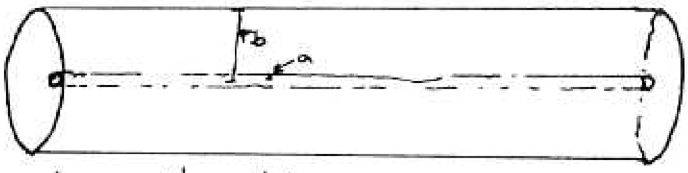
$$\oint B\cdot d\ell = \frac{A}{c}\,\frac{\cos(kz-\omega t)}{s}\,2\pi s = \mu_0 I_{encl}$$

$$\Rightarrow \boxed{I(z,t) = \frac{2\pi A}{\mu_0 c}\cos(kz-\omega t)}$$

The charge and current on the outer conductor are the $\boxed{\text{opposite}}$ of these since the is no E or B inside the metal.

#7 From the previous problem we have the following results

$$\bar{E} = \frac{A\cos(kz-\omega t)}{s}\,\hat{s}$$



A coaxial cable

$$\bar{B} = \frac{A\cos(kz-\omega t)}{cs}\,\hat{\phi}$$

$$\lambda = 2\pi\epsilon_0 A\cos(kz-\omega t), \qquad I = \frac{2\pi A}{\mu_0 c}\cos(kz-\omega t)$$

$$\Delta V = -\int_b^a \bar{E}\cdot d\bar{l} = -\int_b^a \frac{A}{s}\cos(kz-\omega t)\,\hat{s}\cdot d\bar{s}$$

$$= -A\cos(kz-\omega t)\int_b^a \frac{ds}{s} = A\cos(kz-\omega t)\ln\frac{b}{a}$$

Thus, $$Z_0 \equiv \frac{\Delta V}{I} = \frac{A\cos(kz-\omega t)\ln b/a}{\frac{2\pi A}{\mu_0 c}\cos(kz-\omega t)} = \frac{1}{2\pi}\frac{1}{\mu_0 c}\ln\frac{b}{a}$$

$$\boxed{Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\frac{b}{a}}$$

Now,
$$W = \frac{1}{2}LI^2 = \frac{1}{2\mu_0}\int B^2 d\tau = \frac{1}{2\mu_0}\int_a^b \frac{A^2\cos^2(kz-\omega t)}{c^2 s^{\cancel{2}}}\,2\pi \cancel{s}\, l\, ds$$

$$= \frac{1}{2\mu_0}\frac{A^2\cos^2(kz-\omega t)}{c^2}\,2\pi l\ln\frac{b}{a}$$

$$L = \frac{1}{2\mu_0}\frac{A^2\cos^2(kz-\omega t)}{\cancel{c^2}}\,2\pi l\ln b/a\cdot 2\Big/\left(\frac{2\pi A}{\mu_0 \cancel{c}}\cos(kz-\omega t)\right)^2$$

$$= \frac{1}{2\mu_0}\frac{1}{2\pi}\mu_0^2\, l\ln b/a\; 2$$

$$= \frac{\mu_0\ln b/a}{2\pi}\,l \Longrightarrow \boxed{L' = \frac{\mu_0\ln b/a}{2\pi}}$$

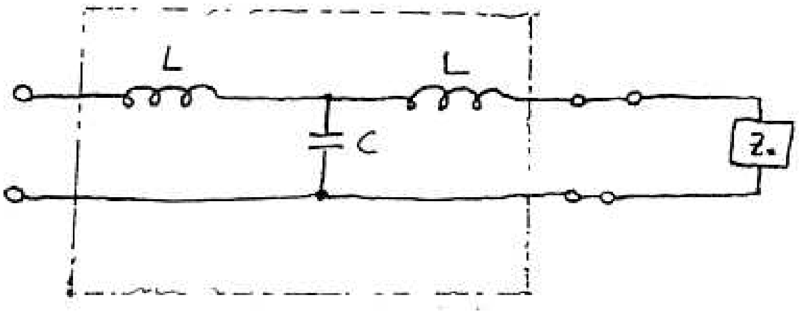
$$C' = \frac{Q}{\Delta V}\frac{1}{\ell} = \frac{\lambda}{\Delta V} = \frac{2\pi\epsilon_0 A\cos(kz-\omega t)}{A\cos(kz-\omega t)\ln b/a} = \boxed{\frac{2\pi\epsilon_0}{\ln b/a}}$$

Thus,

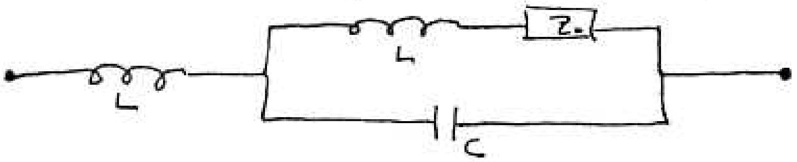
$$Z_0 = \sqrt{\frac{L'}{C'}} = \sqrt{\frac{\mu_0 \ln b/a}{2\pi} \Big/ \frac{2\pi\epsilon_0}{\ln b/a}} = \boxed{\sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{\ln b/a}{2\pi}}$$

#8

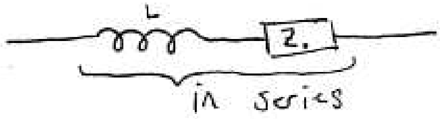
a)



The above circuit is equivalent to the following circuit,

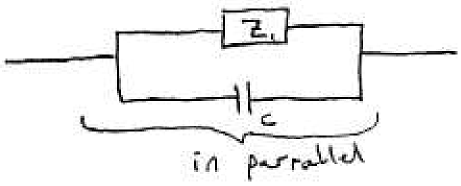


Now, we wish to find the effective impedence of the above arrangement



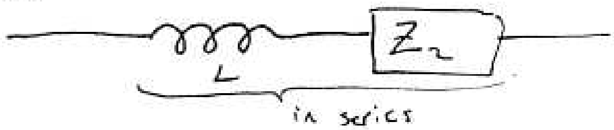
$$\Rightarrow \quad Z_1 = Z_0 + Z_L$$

Next,



$$\Rightarrow \quad Z_2 = \frac{Z_1 Z_c}{Z_1 + Z_c}$$

Finally,



$$\Rightarrow \quad Z_{effective} = Z_L + Z_2$$

So, we have for the effective impedence of the above circuit,

① $$Z_{eff} = Z_L + (Z_0 + Z_L) Z_c \big/ (Z_0 + Z_L + Z_c)$$

The reactance of the capacitor and the inductor are give by.

$Z_L = i\omega L$ (reactance of the inductor)

$Z_c = 1/i\omega C$ (reactance of the capacitor)

If the resulting impedence between the left terminals is to equal Z_0 then we have the following for eq 1,

$$Z_0 = Z_L + (Z_0 + Z_L) Z_c \Big/ (Z_0 + Z_L + Z_c)$$

$$Z_0 (Z_0 + Z_L + Z_c) = Z_L (Z_0 + Z_L + Z_c) + (Z_0 + Z_L) Z_c$$

$$Z_0^2 + \cancel{Z_0 Z_L} + \cancel{Z_0 Z_c} = \cancel{Z_L Z_0} + Z_L^2 + Z_L Z_c + \cancel{Z_0 Z_c} + Z_L Z_c$$

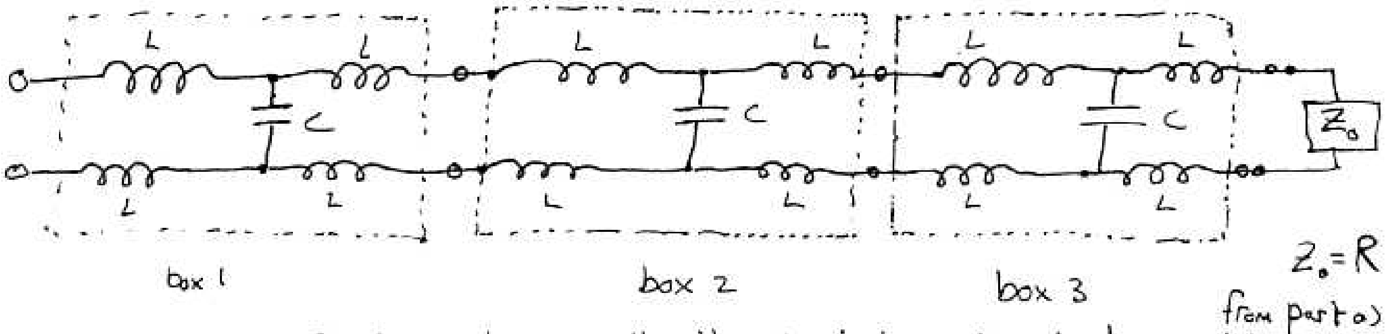
$$Z_0^2 = Z_L^2 + 2 Z_L Z_c$$

$$Z_0 = \sqrt{Z_L^2 + Z_L Z_c} = \sqrt{-\omega^2 L^2 + 2L/C}$$

$$\boxed{Z_0 = \sqrt{\frac{2L}{C} - \omega^2 L^2}}$$

Z_0 is a real number when $\frac{2L}{C} - \omega^2 L^2 > 0 \Rightarrow \omega^2 < \frac{2L}{C}$

b) Now, we wish to add a bunch of these circuits in series



If we add a chain of these boxes with the last box terminating with resistance Z_0 we will have for the last box on the right

- box 3 (Last box in chain) impedence on equals Z_0, thus impedence on left equals Z_0 (from result in part a)

So, • box 2 has impedence Z_0 on the right, thus the impedence on the left equals Z_0 (using result in part a)

Lastly,

• box 1 has impedence Z_0 on the right, thus the impedence on the left equals Z_0 (using result in part a)

So, we have the result that the $\boxed{\text{input impedence equals equals } Z_0}$

• By induction we can extend this to an infinitely long chain.

c) When, $\omega = \sqrt{\frac{2}{LC}}$

$$\Rightarrow \quad Z_0 = \sqrt{\frac{2L}{C} - \frac{2L^{\cancel{2}}}{\cancel{L}C}} = 0$$

OR,

$$\boxed{Z_0 = 0 \quad \text{when} \quad \omega = \sqrt{\frac{2}{LC}}}$$

University of California, Berkeley
Physics 110B Spring 2001   Section 1 (*Strovink*)

Problem Set 3

1. Griffiths 10.3.

2. Griffiths 10.5.

3. Griffiths 10.7.

4. Griffiths 10.10.

5. Griffiths 10.13.

6. Griffiths 10.14.

7. Griffiths 10.20.

8. Consider two electrons each traveling with constant velocity $\beta c\hat{z}$ in the $\hat{z}$ direction, separated by a distance $\hat{x}b$ perpendicular to the $\hat{z}$ direction.
(**a**) Working in the electrons' mutual rest (*) frame, find the force F_x^* with which one electron repels the other.
(**b**) Using the fact that $\Delta p_x^* = \Delta p_x$ is a Lorentz invariant, but $\Delta t^* = \sqrt{1-\beta^2}\Delta t$ is not, find the force F_x of repulsion between the two electrons as evaluated in the lab frame.
(**c**) As an alternative to the approach (a)+(b), work directly in the lab frame. Using Griffiths Eqs. (10.65-10.66), evaluate the electromagnetic fields created by one electron at the position of the other. Use these fields to evaluate the force of mutual repulsion, and compare your answer to (b).

Physics 110B
Homework Solution #3

#1 (Griffiths 10.3)

$$\bar{E} = -\bar{\nabla} V - \frac{\partial \bar{A}}{\partial t} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}} \qquad \bar{B} = \bar{\nabla}\times\bar{A} = \boxed{0}$$

$\bar{A}$ is a vector in the radial direction, therefore it has no curl.

#2 (Griffiths 10.5)

(eq 10.7) $$V' = V - \frac{\partial \lambda}{\partial t} = -\left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right) = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r}}$$

(eq 10.7) $$\bar{A}' = \bar{A} + \bar{\nabla}\lambda = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r} + \left(-\frac{1}{4\pi\epsilon_0}qt\right)\left(-\frac{1}{r^2}\hat{r}\right) = \boxed{0}$$

This gauge function transforms the above nonstandard potentials to the standard potentials that we are used to.

#3 (Griffiths 10.7)

Let's begin with a divergence of $\bar{A}$ that does not satisfy the Lorentz gauge, and then we'll show that we are always able to find a gauge transformation which allows us to satisfy the Lorentz gauge:

$$\bar{\nabla}\cdot\bar{A} \neq -\mu_0\epsilon_0\frac{\partial V}{\partial t} \quad \Rightarrow \quad \bar{\nabla}\cdot\bar{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = B \quad \text{a known fnc}$$

Pick now $\bar{A}'$ and V' to satisfy the Lorentz gauge,

$$\bar{\nabla}\cdot\bar{A}' = -\mu_0\epsilon_0\frac{\partial V'}{\partial t}$$

Thus,

$$\bar{\nabla}\cdot\bar{A}' + \mu_0\epsilon_0 \frac{\partial V'}{\partial t} = \bar{\nabla}\cdot\bar{A} + \nabla^2\lambda + \mu_0\epsilon_0\frac{\partial V}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2}$$

using (eq 10.7)

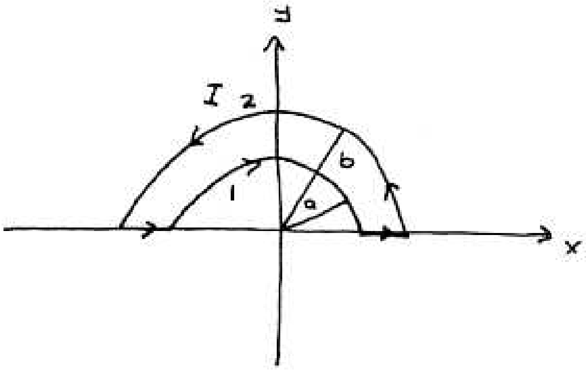
$$= \left(\bar{\nabla}\cdot\bar{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) + \left(\nabla^2\lambda - \mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2}\right) = B + \Box^2\lambda$$

$\Longrightarrow$ • This will equal zero, and thus the Lorentz gauge, if we pick for λ the solution to $\boxed{\Box^2\lambda = -B}$, which we in fact know how to do.

• We can always find a gauge in which $V'=0$, by picking $\boxed{\lambda = \int_0^t V dt'}$

• We cannot in general pick $\bar{A}=0$ since this would make $\bar{B}=0$

#4 (Griffiths 10.10)



$$\bar{A} = \frac{\mu_0}{4\pi}\int \frac{I(t_r)}{r}\, d\bar{l} \quad \text{(eq 10.19)}$$

$$= \frac{\mu_0 k}{4\pi}\int \frac{(t - r/c)}{r}\, d\bar{l}$$

$$= \frac{\mu_0 k}{4\pi}\left(t\oint \frac{d\bar{l}}{r} - \frac{1}{c}\cancelto{0}{\oint d\bar{l}}\right) = \frac{\mu_0 k t}{4\pi}\left(\frac{1}{a}\int_1 d\bar{l} + \frac{1}{b}\int_2 d\bar{l} + 2\hat{x}\int_a^b \frac{dx}{x}\right)$$

$$= \frac{\mu_0 k t}{4\pi}\left(\frac{1}{a}2a + \frac{1}{b}(-2b) + 2\ln\frac{b}{a}\right)\hat{x}$$

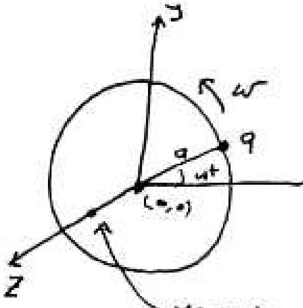
$$\boxed{\bar{A} = \frac{\mu_0 k t}{2\pi}\ln\frac{b}{a}\,\hat{x}}$$

• The changing magnetic field induces the Electric field.

$$\bar{E} = -\frac{\partial\bar{A}}{\partial t} = \boxed{-\frac{\mu_0 k}{2\pi}\ln\frac{b}{a}\,\hat{x}}$$

• Since we only know $\bar{A}$ at one point the center we cannot compute $\bar{\nabla}\times\bar{A}$ to get $\bar{B}$

#5 (Griffiths 10.13)



$t=0,\ (a,0)$

$$\bar{w}(t) = (a\cos\omega t,\ a\sin\omega t)$$

$$\bar{v}(t) = (-a\omega\sin\omega t,\ a\omega\cos\omega t)$$

$$\bar{\mathscr{r}} = \bar{r} - \bar{w}(t_r)$$

$$= z\hat{z} - (a\cos\omega t\,\hat{x} + a\sin\omega t\,\hat{y})$$

$$\mathscr{r}^2 = z^2 + a^2\cos^2\omega t + a^2\sin^2\omega t = z^2 + a^2$$

$$\Rightarrow \mathscr{r} = \sqrt{z^2+a^2}$$

$$\bar{\mathscr{r}}\cdot\bar{v} = -a^2\omega(-\sin\omega t_r\cos\omega t_r + \sin\omega t_r\cos\omega t_r) = 0$$

Thus,

$$V(\bar{r},t) = \underset{\text{(eq 10.39)}}{\frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \bar{\mathscr{r}}\cdot\bar{v})}} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{z^2+a^2}}}$$

$$\text{(eq 10.40)}\quad \bar{A}(\bar{r},t) = \frac{\bar{v}}{c^2}V(\bar{r},t) = \boxed{\frac{1}{4\pi\epsilon_0 c^2}\frac{q\omega a}{\sqrt{z^2+a^2}}(-\sin(\omega t_r)\hat{x} + \cos(\omega t_r)\hat{y})}$$

where, $\boxed{t_r = t - \dfrac{\sqrt{z^2+a^2}}{c}}$

#6 (Griffiths 10.14)

$$\text{(eq 10.42)}\quad V(\bar{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c^2t - \bar{r}\cdot\bar{v})^2 + (c^2-v^2)(r^2-c^2t^2)}}$$

Evaluating the term under the square root we have,

$$S = \cancel{c^4t^2} - 2c^2t(\bar{r}\cdot\bar{v}) + (\bar{r}\cdot\bar{v})^2 + c^2r^2 - \cancel{c^4t^2} - v^2r^2 + v^2c^2t^2$$

$$= (\bar{r}\cdot\bar{v})^2 + (c^2-v^2)r^2 + c^2(vt)^2 - 2c^2(\bar{r}\cdot\bar{v}t)$$

$$= (\bar{r}\cdot\bar{v})^2 + (c^2-v^2)r^2 + c^2(r^2+R^2-2\bar{r}\cdot\bar{R}) - 2c^2(r^2-\bar{r}\cdot\bar{R})$$

$\bar{R} \equiv \bar{r}-\bar{v}t$

$\bar{v}t = \bar{r}-\bar{R}$

① $$= (\bar{r}\cdot\bar{v})^2 - r^2v^2 + c^2R^2$$

But, we have

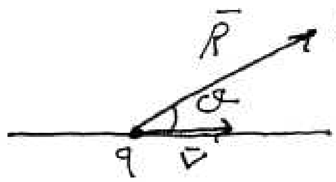
$$(\bar{r}\cdot\bar{v})^2 - r^2v^2 = ((\bar{R}+\bar{v}t)\cdot\bar{v})^2 - (R+vt)^2v^2$$

$$= (\bar{R}\cdot\bar{v})^2 + \cancel{v^4t^2} + \cancel{2(\bar{R}\cdot\bar{v})v^2t} - R^2v^2 - \cancel{2(\bar{R}\cdot\bar{v})tv^2} - \cancel{v^2t^2v^2}$$

$$= (\bar{R}\cdot\bar{v})^2 - R^2v^2 = R^2v^2\cos^2\theta - R^2v^2$$

$$= -R^2v^2(1-\cos^2\theta)$$

$$= -R^2v^2\sin^2\theta$$



Plugging this result into step ①,

$$S = -R^2v^2\sin^2\theta + c^2R^2$$

$$= c^2R^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)$$

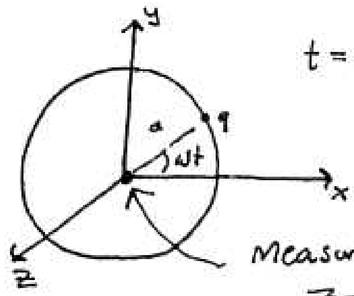
Plugging this into the equation for potential V,

$$\boxed{V(r,t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}}$$

#7 (Griffiths 10.20)

$$\bar{w}(t) = R(\cos\omega t, \sin\omega t)$$

$$\bar{v}(t) = R\omega(-\sin\omega t, \cos\omega t)$$



$t=0$, $(a,0)$

Measuring field at $z=0$

$$\bar{a}(t) = -R\omega^2(\cos\omega t, \sin\omega t) = -\omega^2 w(t)$$

$$\bar{\mathscr{r}} = \bar{r} - \bar{w}(t_r) = -\bar{w}(t), \qquad \bar{u} = c\hat{\mathscr{r}} - v(t_r)$$

$$|\bar{\mathscr{r}}| = |\bar{w}(t)| = R, \qquad t_r = t - R/c$$

(eq 10.65) $\bar{E}(\bar{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\bar{\mathscr{r}}\cdot\bar{u})^3}\left((c^2-v^2)\bar{u} + \bar{\mathscr{r}}\times(\bar{u}\times\bar{a})\right)$

Solving the individual parts of the above equation,

$$\bar{u} = -c(\cos\omega t_r\,\hat{x} + \sin\omega t_r\,\hat{y}) - \omega R(-\sin\omega t_r\,\hat{x} + \cos\omega t_r\,\hat{y})$$

$$= -(c\cos\omega t_r - \omega R\sin\omega t_r)\hat{x} - (c\sin\omega t_r + \omega R\cos\omega t_r)\hat{y}$$

$$\bar{\mathscr{r}}\times(\bar{u}\times\bar{a}) = (\bar{\mathscr{r}}\cdot\bar{a})\bar{u} - (\bar{\mathscr{r}}\cdot\bar{u})\bar{a}$$

$$\rightarrow \bar{\mathscr{r}}\cdot\bar{a} = -\bar{w}\cdot(-\omega^2\bar{w}) = \omega^2R^2$$

$$\rightarrow \bar{\mathscr{r}}\cdot\bar{u} = R(c\cos^2\omega t_r - \cancel{\omega R\sin\omega t_r\cos\omega t_r} + c\sin^2\omega t_r + \omega R\sin\omega t_r\cos\omega t_r)$$

$$= Rc$$

$$\rightarrow v^2 = (\omega R)^2$$

Plugging these results into (eq 10.65),

$$\bar{E}(r,t) = \frac{q}{4\pi\epsilon_0}\frac{R}{(Rc)^3}\left((c^2-v^2)\bar{u} + \omega^2R^2\bar{u} - Rc(-\omega^2\bar{w})\right)$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{R^2c^3}\left((c^2 - \cancel{\omega^2R^2})\bar{u} + \cancel{\omega^2R^2\bar{u}} + Rc\omega^2\bar{w}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{R^2c^2}\Big(-c(c\cos\omega t_r - \omega R\sin\omega t_r)\hat{x} - c(c\sin\omega t_r + \omega R\cos\omega t_r)\hat{y}$$
$$+ R^2\omega^2(\cos\omega t_r\,\hat{x} + \sin\omega t_r\,\hat{y})\Big)$$

$$\boxed{\bar{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{(Rc)^2}\Big(\big((\omega^2R^2-c^2)\cos\omega t_r + \omega Rc\sin\omega t_r\big)\hat{x} + \big((\omega^2R^2-c^2)\sin\omega t_r - \omega Rc\cos\omega t_r\big)\hat{y}\Big)}$$

(eq 10.66) $\bar{B}(\bar{r},t) = \frac{1}{c}\hat{\boldsymbol{\imath}}\times\bar{E}(r,t) = \frac{1}{c}(\hat{\boldsymbol{\imath}}_x E_y - \hat{\boldsymbol{\imath}}_y E_x)\hat{z}$

$$= -\frac{1}{c}\frac{q}{4\pi\epsilon_0}\frac{1}{(Rc)^2}\Big(\cos\omega t_r\big(\cancel{(\omega^2R^2-c^2)\sin\omega t_r} - \omega Rc\cos\omega t_r\big) - \sin\omega t_r\big(\cancel{(\omega^2R^2-c^2)\cos\omega t_r} + \omega Rc\sin\omega t_r\big)\Big)\hat{z}$$

$$= -\frac{q}{4\pi\epsilon_0}\frac{1}{R^2c^3}\left(-\omega Rc\cos^2\omega t_r - \omega Rc\sin^2\omega t_r\right)\hat{z}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{R^2c^3}\omega Rc\,\hat{z} = \boxed{\frac{q}{4\pi\epsilon_0}\frac{\omega}{Rc^2}\hat{z}}$$

To obtain the field at the center of a ring in terms of current we have

$q \to \lambda 2\pi R$ and $I = \lambda v = \lambda\omega R$

So, $q = \frac{I}{\omega R}2\pi R = \frac{2\pi I}{\omega}$

Thus, $\bar{B} = \frac{\cancel{2\pi} I}{\cancel{\omega}}\frac{1}{\cancel{4\pi}_2\epsilon_0}\frac{\cancel{\omega}}{Rc^2}\hat{z} = \boxed{\frac{\mu_0 I}{2R}\hat{z}}$

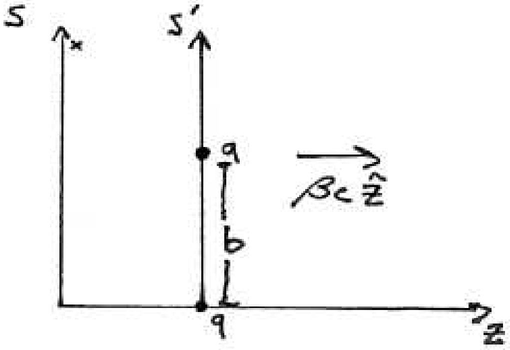
This is the same result as in Exercise 5.6

#8

a) rest frame S',

S - Lab frame
S' - Rest frame

$$\boxed{F_x' = \frac{q^2}{4\pi\epsilon_0 b^2}\hat{x}}$$



b) In the lab frame S,

$$F_x = \frac{dp_x}{dt} = \frac{dp_x'}{\gamma(dt' + v/c^2 dx')}$$

by doing a Lorentz transformation on the numerator and denominator

$$= \frac{dp_x'}{dt'} \Big/ \gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right) = F_x' \Big/ \gamma\left(1 + \frac{v}{c^2}\cancel{u_x'}^{\,0}\right)$$

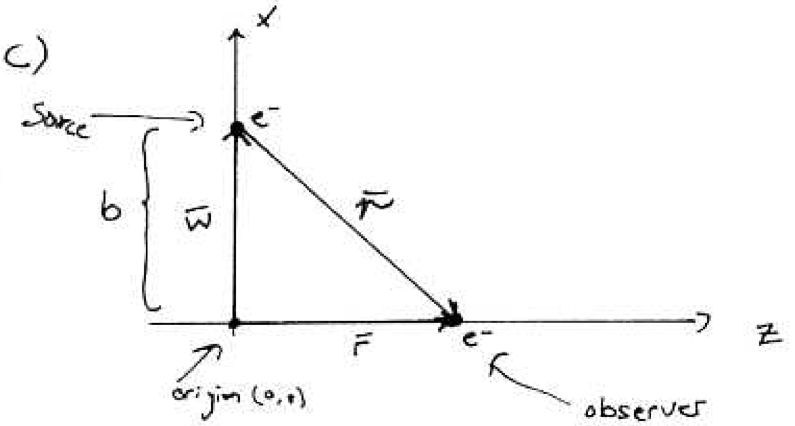
$$= F_x'/\gamma$$

$\beta \equiv \frac{v}{c}$

$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$

$$\Rightarrow \boxed{F_x = \frac{q^2}{4\pi\epsilon_0 b^2}\sqrt{1-\beta^2}}$$

c)



$\bar{w}$ - distance of source from origin

$\bar{𝓇}$ - distance from source to observer

$\bar{r}$ - distance from origin to observer

From Griffiths we have the following equations,

- $\bar{E}(\bar{r},t) = \frac{q}{4\pi\epsilon_0}\frac{𝓇}{(\bar{𝓇}\cdot\bar{u})^3}\left((c^2-v^2)\bar{u} + \bar{𝓇}\times(\bar{u}\times\bar{a})\right)$ (eq 10.65)

- $\bar{B}(\bar{r},t) = \frac{1}{c}\hat{𝓇}\times\bar{E}(\bar{r},t)$ (eq 10.66)

We will need to now the varius quantities in these equations in order to solve them,

$$\bar{w}(t) = b\hat{x} + \beta c t\,\hat{z}$$

$\beta \equiv \frac{v}{c}$

$$\bar{v}(t) = \frac{d\bar{w}}{dt} = \boxed{\beta c\,\hat{z} = \bar{v}}$$

$\bar{a}(t) = 0$

$$\bar{r} = \left(v\left(\tfrac{\mathscr{r}}{c}\right) + \beta c t\right)\hat{z} = \beta(\mathscr{r} + ct)\hat{z}$$

$$\bar{\mathscr{r}} = \bar{r} - \bar{w} = \beta(\mathscr{r} + ct)\hat{z} - b\hat{x} - \beta c t\hat{z} = \boxed{\beta\mathscr{r}\hat{z} - b\hat{x} = \bar{\mathscr{r}}}$$

$$|\bar{\mathscr{r}}| = \sqrt{\beta^2\mathscr{r}^2 + b^2}$$

$$\Rightarrow \mathscr{r}^2 = \beta^2\mathscr{r}^2 + b^2 \quad \Rightarrow \mathscr{r}^2(1-\beta^2) = b^2$$

$$\Rightarrow \boxed{\mathscr{r} = (1-\beta^2)^{-1/2} b}$$

$$\bar{u} = c\hat{\mathscr{r}} - \bar{v} = c\bar{\mathscr{r}}/\mathscr{r} - \bar{v} = c\left(\beta\hat{z} - \tfrac{b}{\mathscr{r}}\hat{x} - \beta\hat{z}\right)$$

$$= -\frac{cb}{\mathscr{r}}\hat{x} = -\frac{c\cancel{b}}{(1-\beta^2)^{-1/2}\cancel{b}}\hat{x} = -c(1-\beta^2)^{1/2}\hat{x}$$

$$\Rightarrow \boxed{\bar{u} = -c(1-\beta^2)^{1/2}\hat{x}}$$

Now, we begin to solve the individual pieces of (eq 10.65),

$$\bar{\mathscr{r}}\cdot\bar{u} = (1-\beta^2)^{1/2} cb$$

$$\bar{\mathscr{r}}\times(\bar{u}\times\cancelto{0}{\bar{a}}) = 0$$

Now, we can put all these results into the equation for the Electric field (eq 10.65):

$$\bar{E}(\bar{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\bar{\mathscr{r}}\cdot\bar{u})^3}\left((c^2-v^2)\bar{u} + \bar{\mathscr{r}}\times(\bar{u}\times\cancelto{0}{\bar{a}})\right)$$

$$= \frac{q}{4\pi\epsilon_0}\frac{(1-\beta^2)^{-1/2}\cancel{b}}{(1-\beta^2)^{3/2}c^3 b^{\cancel{3}2}}\left(-(c^2-\beta^2c^2)(1-\beta^2)^{1/2}_c\hat{x}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{(1-\beta^2)^2 c^3 b^2}\left(-c^3(1-\beta^2)^{3/2}\hat{x}\right)$$

$$= -\frac{q}{4\pi\epsilon_0}\ \frac{}{(1-\beta^2)^{1/2}\, b^2}\ \hat{x}$$

$$\Rightarrow \boxed{\bar{E} = -\frac{1}{4\pi\epsilon_0}\ \frac{q}{(1-\beta^2)^{1/2}\, b^2}\ \hat{x}}$$

Now, we solve for the total force using (eq 10.67)

$$\bar{F} = q\,(\bar{E} + \bar{v} \times \bar{B})$$

$$= q\left(\bar{E} + \bar{v} \times \left(\tfrac{1}{c}\hat{r} \times \bar{E}\right)\right)$$

$$= -\frac{1}{4\pi\epsilon_0}\ \frac{q^2}{(1-\beta^2)^{1/2}\, b^2}\left(\hat{x} + \beta c\hat{z} \times \left(\tfrac{1}{c}(\beta\hat{z} - \tfrac{b}{r}\hat{x}) \times \hat{x}\right)\right)$$

$$= -\frac{1}{4\pi\epsilon_0}\ \frac{q^2}{(1-\beta^2)^{1/2}\, b^2}\left(\hat{x} + \beta^2\, \hat{z} \times \hat{y}\right)$$

$$= -\frac{1}{4\pi\epsilon_0}\ \frac{q^2}{(1-\beta^2)^{1/2}\, b^2}\,(1-\beta^2)\ \hat{x}$$

$$\Rightarrow \boxed{\bar{F}_x = -\frac{1}{4\pi\epsilon_0}\ \frac{q^2}{b^2}\ \sqrt{1-\beta^2}\ \hat{x}}$$

- Which is the same result that we got in part a)

University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

Problem Set 4

1. Griffiths 12.6.

2. Griffiths 12.18.

3. Griffiths 12.19.

4. Griffiths 12.20.

5. Griffiths 12.32.

6. Inertial reference frames $\mathcal{S}'$ and $\mathcal{S}$ coincide at $t' = t = 0$. You may ignore the z dimension, so that a point in spacetime is determined by only three quantities $r \equiv (ct, x, y)$. The Lorentz transformation between $\mathcal{S}$ and $\mathcal{S}'$ is given by

$$\begin{pmatrix} ct' \\ x' \\ y' \end{pmatrix} = \mathcal{L} \begin{pmatrix} ct \\ x \\ y \end{pmatrix} ,$$

where $\mathcal{L}$ is a 3×3 matrix.
(**a**) Assume for this part that $\mathcal{S}'$ moves with velocity

$$\mathbf{V} = \beta c \hat{\mathbf{x}}$$

with respect to $\mathcal{S}$. Using your knowledge of Lorentz transformations (no derivation necessary), write $\mathcal{L}$ for this case.
(**b**) Assume for this part that $\mathcal{S}'$ moves with velocity

$$\mathbf{V} = \beta c \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}}$$

with respect to $\mathcal{S}$. Find $\mathcal{L}$ for this case. (*Hint.* Rotate to a system in which $\mathbf{V}$ is along the $\hat{\mathbf{x}}$ axis, transform using your answer for part (**a.**), and then rotate back. Check that your result is symmetric under interchange of x and y, as is $\mathbf{V}$, and that it reduces to the unit matrix as $\beta \to 0$.)

7. The now retired Bevatron at Berkeley Lab is famous for having produced the first observed antiprotons (you may have glimpsed white-maned Nobelist Owen Chamberlain, one of the first observers, being helped to his seat at Physics Department colloquia). An economical reaction for producing antiprotons is

$$p + p \to p + p + p + \bar{p} ,$$

where the first proton is part of a beam, the second is at rest in a target, and $\bar{p}$ is an antiproton. Because of the CPT theorem, both p and $\bar{p}$ must have the same mass ($= 0.94 \times 10^9$ eV).

At threshold, all four final state particles have essentially zero velocity *with respect to each other*. What is the beam energy in that case? (The actual Bevatron beam energy was 6×10^9 eV).

8. (Taylor and Wheeler problem 51)
The clock paradox, version 3.
Can one go to a point 7000 light years away – *and return* – without aging more than 40 years? "Yes" is the conclusion reached by an engineer on the staff of a large aviation firm in a recent report. In his analysis the traveler experiences a constant "1 g" acceleration (or deceleration, depending on the stage reached in her journey). Assuming this limitation, is the engineer right in his conclusion? (For simplicity, limit attention to the first phase of the motion, during which the astronaut accelerates for 10 years – then double the distance covered in that time to find how far it is to the most remote point reached in the course of the journey.)
(**a**)
The acceleration is *not* $g = 9.8$ meters per second per second relative to the laboratory frame. If it were, how many times faster than light would the spaceship be moving at the end of ten years (1 year $= 31.6 \times 10^6$ seconds)? *If the acceleration is not specified with respect to the laboratory, then with respect to what is it specified?* Discussion: Look at the bathroom scales on which one is standing! The rocket jet is always turned up to the point where these scales read one's *correct* weight. Under these conditions one is being accelerated at 9.8 meters per second per second with respect to a spaceship

that (1) instantaneously happens to be riding alongside with identical velocity, but (2) is *not* being accelerated, and, therefore (3) *provides the* (momentary) *inertial frame of reference relative to which the acceleration is* g.

(**b**)

How much velocity does the spaceship have after a given time? This is the moment to object to the question and to rephrase it. *Velocity* βc is not the simple quantity to analyze. The simple quantity is the *boost parameter* η. This parameter is simple because it is *additive* in this sense: Let the boost parameter of the spaceship with respect to the imaginary instantaneously comoving inertial frame change from 0 to $d\eta$ in an astronaut time $d\tau$. Then the boost parameter of the spaceship with respect to the *laboratory* frame changes in the same astronaut time from its initial value η to the subsequent value $\eta + d\eta$. Now relate $d\eta$ to the acceleration g in the instantaneously comoving inertial frame. In this frame $g\,d\tau = c\,d\beta = c\,d(\tanh\eta) = (c/\cosh^2(\eta \approx 0))\,d\eta \approx c\,d\eta$ so that

$$c\,d\eta = g\,d\tau$$

Each lapse of time $d\tau$ on the astronaut's watch is accompanied by an additional increase $d\eta = \frac{g}{c}d\tau$ in the boost parameter of the spaceship. In the laboratory frame the total boost parameter of the spaceship is simply the sum of these additional increases in the boost parameter. Assume that the spaceship starts from rest. Then its boost parameter will increase linearly with *astronaut* time according to the equation

$$c\eta = g\tau$$

This expression gives the boost parameter η of the spaceship in the *laboratory* frame at any time τ in the *astronaut's* frame.

(**c**)

What laboratory distance x does the spaceship cover in a given astronaut time τ? At any instant the velocity of the spaceship in the laboratory frame is related to its boost parameter by the equation $dx/dt = c\tanh\eta$ so that the distance dx covered in *laboratory* time dt is

$$dx = c\,\tanh\eta\,dt$$

Remember that the time between ticks of the astronaut's watch $d\tau$ appear to have the larger value dt in the laboratory frame (time dilation) given by the expression

$$dt = \cosh\eta\,d\tau$$

Hence the laboratory distance dx covered in *astronaut* time $d\tau$ is

$$dx = c\,\tanh\eta\,\cosh\eta\,d\tau = c\,\sinh\eta\,d\tau$$

Use the expression $c\eta = g\tau$ from part b to obtain

$$dx = c\,\sinh\left(\frac{g\tau}{c}\right)d\tau$$

Sum (integrate) all these small displacements dx from zero astronaut time to a final astronaut time to find

$$x = \frac{c^2}{g}\left[\cosh\left(\frac{g\tau}{c}\right) - 1\right]$$

This expression gives the laboratory *distance* x covered by the spaceship at any time τ in the astronaut's frame.

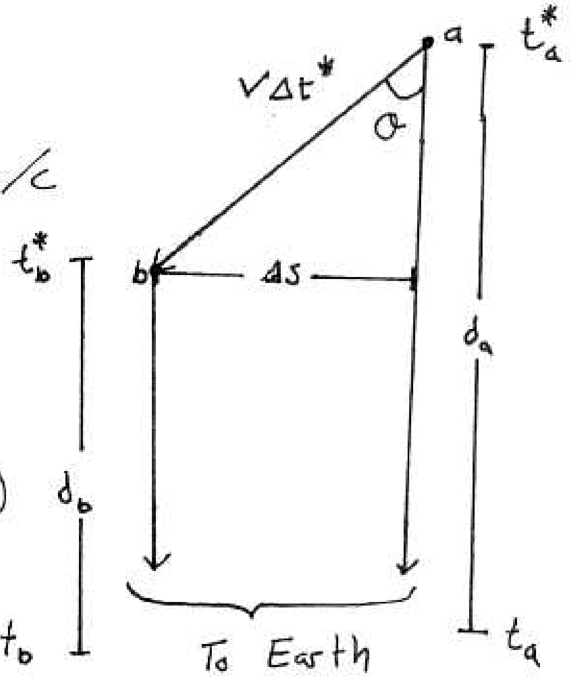
(**d**)

Plugging in the appropriate numerical values, determine whether the engineer is correct in his conclusion reported at the beginning of this exercise.

Physics 110B
Homework #4

#1 (Griffiths 12.6)

- Light signal leaves a at time t_a^*; arrives at Earth at time $t_a = t_a' + d_a/c$
- Light signal leaves b at time t_b^*; arrives at Earth at time $t_b = t_b' + d_b/c$



Thus,

$$\Delta t = t_b - t_a = (t_b^* + d_b/c) - (t_a^* + d_a/c)$$
$$= t_b^* - t_a^* + (d_b - d_a)/c$$
$$= \Delta t^* + (-v\Delta t' \cos\theta)/c$$
$$= \Delta t^*(1 - v/c \cos\theta)$$

d_a is the distance from a to Earth and d_b is the distance from b to Earth

$$\Delta s = v\Delta t^* \sin\theta = \frac{v \sin\theta \, \Delta t}{(1 - v/c \cos\theta)}$$

Therefore, the apparent velocity is,

$$V_{apparent} = \boxed{\frac{\Delta s}{\Delta t} = \frac{v \sin\theta}{(1 - \frac{v}{c}\cos\theta)}}$$ is the apparent velocity

The maximum value we get by taking the derivative and setting it equal to zero:

$$\frac{dV_{apparent}}{d\theta} = \frac{v((1 - \frac{v}{c}\cos\theta)\cos\theta - \sin\theta \frac{v}{c}\sin\theta)}{(1 - \frac{v}{c}\cos\theta)^2} = 0$$

$$\Rightarrow (1 - \tfrac{v}{c}\cos\theta)\cos\theta = \tfrac{v}{c}\sin^2\theta$$

$$\cos\theta = \frac{v}{c}\left(\cos^2\theta + \sin^2\theta\right)$$

$$\cos\theta = v/c$$

$$\boxed{\theta_{max} = \cos^{-1}(v/c)}$$

At the max angle, $V_{apparent} = \dfrac{v\sqrt{1 - v^2/c^2}}{1 - v^2/c^2} = \dfrac{v}{\sqrt{1 - v^2/c^2}}$

as $v \to c$, $V_{apparent} \to \infty$ even though $v < c$, therefore we see that the apparent velocity can be larger then the speed of light.

#2 (Griffiths 12.18)

a) $$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \boxed{\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\equiv\, M_{Galilean}}} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

b) $$\boxed{\Lambda = \begin{pmatrix} \gamma & 0 & -\gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

c) $$\Lambda = \begin{pmatrix} \gamma' & 0 & -\gamma'\beta' & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma'\beta' & 0 & \gamma' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \gamma\gamma' & -\gamma\gamma'\beta & -\gamma'\beta' & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ -\gamma'\gamma\beta' & \gamma\gamma'\beta\beta' & \gamma' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

- $\boxed{\text{Yes,}}$ the order of applying the matrices does matter. If switched the orders primes and unprimes would be switched giving us a different matrix

#3 (Griffiths 12.19)

a)
$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{(eq 12.24)}$$

$$\tanh\theta \equiv V/c \quad \text{(eq 12.34)}$$

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta}, \quad \cosh^2\theta - \sinh^2\theta = 1$$

$$\gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\tanh^2\theta}} = \frac{\cosh\theta}{\sqrt{\cosh^2\theta - \sinh^2\theta}} = \cosh\theta$$

$$\gamma\beta = \cosh\theta \tanh\theta = \sinh\theta$$

Therefore,

$$\boxed{\Lambda = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

compare with Rotation matrix
$$R = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b) Einstein's velocity addition rule

$$u' = \frac{u - v}{1 - uv/c^2} \quad \text{(eq 12.20)} \Rightarrow \frac{u'}{c} = \frac{u/c - v/c}{1 - uv/c^2}$$

$$\Rightarrow \tanh\phi' = \frac{\tanh\phi - \tanh\theta}{1 - \tanh\phi\tanh\theta}, \quad \text{where } \tanh\phi \equiv u/c, \ \tanh\theta \equiv v/c$$

$\tanh\phi' = \tanh(\phi - \theta)$ using a hyperbolic trig. identity

OR: $\boxed{\phi' = \phi - \theta}$

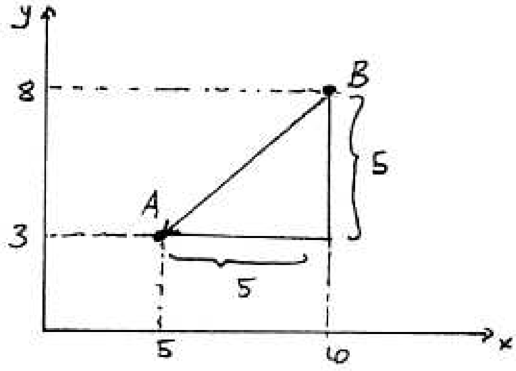
#4 (Griffiths 12.20) (Note: using Prof. Stravink notation w/ (1,-1,-1,-1) signature)

a) i) $I = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = (5-15)^2 - (10-5)^2 - (8-3)^2 - (0-0)^2$

$= 100 - 25 - 25 = \boxed{50}$ timelike

ii) If the events occur simultaneously $\Delta t' = 0$, "I" in this case would be negative, which it isn't. Thus, the events $\boxed{\text{cannot occur simultaneously.}}$

iii) S' travels in the direction from B toward A, making the trip in in time $\Delta t = 10/c$, if the moving frame does this then the two events will occur at the same place though at different times,



$\vec{v} = \dfrac{-5\hat{x} - 5\hat{y}}{10/c} = \boxed{-\frac{c}{2}\hat{x} - \frac{c}{2}\hat{y}}$

We note $v^2/c^2 = 1/2 \Rightarrow v = \frac{1}{\sqrt{2}}c$ which has $v < c$ as has to be

b) i) $I = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z = (3-1)^2 - (5-2)^2 + 0 + 0 = \boxed{-5}$ spacelike

ii) $\boxed{\text{Yes, can occur simultaneously.}}$ We want $\Delta t' = 0$, therefore using the Lorentz transformation,

$c\Delta t' = \gamma(c\Delta t - \beta\Delta x) = 0 \Rightarrow c\Delta t = \beta\Delta x$

Or, $v/c = c\Delta t/\Delta x = \frac{(3-1)}{(5-2)} = \frac{2}{3}$ So, $\boxed{v = \frac{2}{3}c \text{ in the x direction}}$

iii) $\boxed{\text{No, the events cannot occur at the same location,}}$ since this would require that $\Delta x' = \Delta y' = \Delta z' = 0$ so that "I" would be positive which it is not.

#5 (Griffiths 12.32)

$$\overset{\mapsto}{1} + 2 \longrightarrow 3$$

$E_1 = 2mc^2$ $\quad m \quad$ E_3, m_3, p_{3x}

$$p_1 = (2mc, p_{x1}, 0, 0) \qquad p_3 = (E_3, p_{3x}, 0, 0)$$

$$p_2 = (mc, 0, 0, 0)$$

- $$p_1 + p_2 = p_3$$

$$(p_1 + p_2)^2 = p_3^2$$

$$p_1 \cdot p_1 + 2 p_1 \cdot p_2 + p_2 \cdot p_2 = p_3 \cdot p_3$$

$$m^2c^2 + 2(2m^2c^2) + m^2c^2 = m_3^2 c^2$$

$$\boxed{m_3 = \sqrt{6}\, m}$$

- $$p_1 \cdot p_1 = 4m^2c^2 - p_{x1}^2 = m^2c^2 \implies p_{x1} = \sqrt{3}\, mc$$

$$\bar{p}_{T\,initial} = p_{x1} + \cancel{p_{x2}} = \sqrt{3}\, mc$$

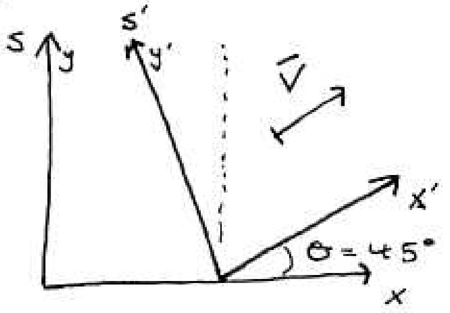
$$E_{T\,initial} = E_1 + E_2 = 3mc^2$$

$$\bar{V}_{cm} = \frac{\bar{p}_{Ti}\, c^2}{E_{Ti}} = \frac{(\sqrt{3}\, mc)c^2}{3mc^2} = \boxed{\frac{1}{\sqrt{3}} c = V_{cm}}$$

6.(a)

$$\boxed{\mathcal{L} = \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

(b)



$$\bar{v} = \frac{\beta c}{\sqrt{2}}(\hat{x} + \hat{y}) = \beta c(\cos 45°, \sin 45°)$$

$$\mathcal{L}_{total} = R^{-1}\mathcal{L}R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \gamma & -\gamma\beta\cos\theta & -\gamma\beta\sin\theta \\ -\gamma\beta & \gamma\cos\theta & \gamma\sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\gamma\beta\cos\theta & -\gamma\beta\sin\theta \\ -\gamma\beta\cos\theta & \gamma\cos^2\theta + \sin^2\theta & \gamma\cos\theta\sin\theta - \sin\theta\cos\theta \\ -\gamma\beta\sin\theta & \gamma\cos\theta\sin\theta \quad \sin\theta\cos\theta & \gamma\sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$\boxed{\mathcal{L}_{Total} = \begin{pmatrix} \gamma & -\gamma\beta/\sqrt{2} & -\gamma\beta/\sqrt{2} \\ -\gamma\beta/\sqrt{2} & (1+\gamma)/2 & (\gamma-1)/2 \\ -\gamma\beta/\sqrt{2} & (\gamma-1)/2 & (1+\gamma)/2 \end{pmatrix}}$$

- as $\beta \to 0$ $\quad \gamma \to 1$

$$\mathcal{L}_{total} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Also, the result is symmetric under the interchange of x and y.

#7.

$$\overset{\mapsto}{p}_1 + p_2 \longrightarrow p + p + p + \bar{p}$$

E_b

final state
moving together uniformly

$$p_1 = (E_b/c,\ p_{1x}, 0, 0) \qquad p'_{FT} = (4m_p c, 0, 0, 0)$$

$$p_2 = (m_p c, 0, 0, 0)$$

$$p_1 + p_2 = p_{\text{final total}}$$

$$(p_1 + p_2)\cdot(p_1 + p_2) = p_{FT}\cdot p_{FT} = p'_{FT}\cdot p'_{FT} = 16 m_p^2 c^2$$

$$p_1\cdot p_1 + 2p_1\cdot p_2 + p_2\cdot p_2 = 16 m_p^2 c^2$$

$$m_p^2 c^2 + 2 m_p E_b + m_p^2 c^2 = 16 m_p^2 c^2$$

$$E_b = 7 m_p c^2 = 7\times(.94\times 10^9\,\text{eV})$$

$$\boxed{E_{\text{beam}} = 6.58\times 10^9\,\text{eV}}$$

#8.

(a) $V = gt = (9.8)\cdot(10)\cdot(31.6\times 10^6) = \boxed{3.10\times 10^9\,\text{m/s}}$

(b) $g\,d\tau = c\,d\beta = c\,d(\tanh\eta) = (c/\cosh^2(\eta\approx 0))\,d\eta \approx c\,d\eta$

$$c\,d\eta = g\,d\tau$$

$$\Rightarrow \int c\,d\eta = \int g\,d\tau \quad\Rightarrow\quad \boxed{c\eta = g\tau}$$

(c) $\beta = \tanh\eta \Rightarrow \frac{dx}{dt} = c\tanh\eta$

$\Rightarrow dx = c\tanh\eta\, dt$

Now, $dt = \gamma\, d\tau = \cosh\eta\, d\tau$

$\Rightarrow dx = c\tanh\eta\cosh\eta\, d\tau = c\sinh\eta\, d\tau$

Using $c\eta = g\tau$

$dx = c\sinh\left(\frac{g\tau}{c}\right) d\tau$

$\int dx = \int c\sinh\left(\frac{g\tau}{c}\right) d\tau$

$\Rightarrow \boxed{x = \frac{c^2}{g}\left(\cosh\left(\frac{g\tau}{c}\right) - 1\right)}$

(d) The maximum amount of time the astronaut can travel is $\tau_M = 40$ yrs. in a quarter of that time she travels $\tau = 10$ yrs.

Now, figuring the x-distance for this time:

$x = \frac{c^2}{g}\left(\cosh\left(\frac{g\tau_{1/4}}{c}\right) - 1\right) = \frac{(2.998\times10^8)^2}{9.8}\left(\cosh\left(\frac{9.8\cdot10\cdot31.6\times10^6}{2.998\times10^8}\right) - 1\right)$

$= 1.404\times10^{20}\,m$

$X_{Total} = 2x = 2.8\times10^{20}\,m \cdot \frac{1\text{ lyr.}}{31.6\times10^6 s\cdot 3\times10^8 m/s}$

$\boxed{X_{Total\ Max} = 29{,}650 \text{ light years}}$

$X_{total\ max\ out} > 7000$ lightyears. Thus, the $\boxed{\text{Engineer was correct.}}$

University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

Problem Set 5

1. A particle with $\gamma\beta = 4/3$ decays into two massless particles with the same energy each.
(**a**) If the parent particle has mean proper life τ, calculate its mean flight path x before decay.
(**b**) Calculate the opening angle ψ between the two daughter particles.

2. Here's an adult version of Griffiths 12.35. In a pair annihilation experiment, a positron (mass m) with total energy $E = \gamma mc^2$ hits an electron (same mass, but opposite charge) at rest. (Griffiths has it the other way around, but that's unrealistic – it's easy to make a positron beam, but hard to make a positron target.) The two particles annihilate, producing two photons. (If only one photon were produced, energy-momentum conservation would force it to be a massive particle travelling at a velocity less than c.) If one of the photons emerges at angle θ relative to the incident positron direction, show that its energy E' is given by

$$\frac{mc^2}{E'} = 1 - \sqrt{\frac{\gamma-1}{\gamma+1}}\cos\theta \ .$$

(In particular, if the photon emerges perpendicular to the beam, its energy is equal to mc^2, independent of the beam energy. Similar results have been used to design clever experiments.)
[*Hint*: Griffiths 12.35 uses "convenient" values for γ and θ, but his solution to this problem is nevertheless full of messy algebra. Instead, as in class, write a four-vector equation expressing energy-momentum conservation, take the dot product of either side with itself, and get a concise result in a few lines.]

3. Griffiths 12.44.

4. Griffiths 12.45.

5. Griffiths 12.46.

6. A particle travelling with velocity $\beta c\hat{x}$ has a property represented by the contravariant four-vector h^μ. It is known that $p_\mu h^\mu = 0$, where p_μ is the particle's covariant four-momentum, where, by convention, repeated Greek indices are summed from 0 to 3. Write the components of h^μ in the laboratory as a function of those components in the particle's rest frame which are nonzero.

7. The metric tensor $g_{\mu\nu}$ is defined by

$$h_\mu \equiv g_{\mu\nu}h^\nu \ ,$$

where h^μ and h_μ are the contravariant and covariant versions of the four-vector h, whose invariant length^2 is equal to $h_\mu h^\mu$.
(**a**) Write out the elements of $g_{\mu\nu}$ (in flat spacetime, to which special relativity is pertinent).
(**b**) A contravariant four-tensor $T^{\mu\nu}$ is transformed to its covariant version $T_{\mu\nu}$ by two metric tensor multiplications:

$$T_{\mu\nu} \equiv g_{\mu\alpha}T^{\alpha\beta}g_{\beta\nu} \ .$$

Show that

$$g_{\mu\nu} = g^{\mu\nu} \ .$$

(**c**) Show that

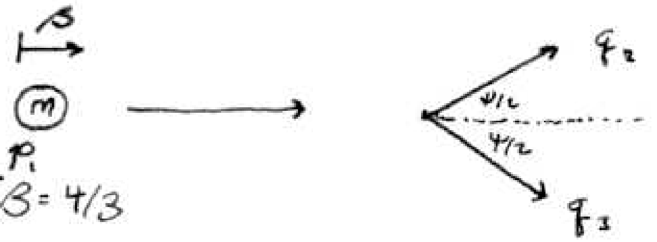
$$g_{\mu\alpha}g^{\alpha\nu} = \delta^\nu_\mu \ ,$$

where the 4-dimensional Kronecker delta function satisfies $\delta^\nu_\mu = 0$ for $\mu \neq \nu$ and $\delta^\mu_\mu = 1$ for $0 \le \mu \le 3$.

8. Consider the antisymmetric contravariant tensor $H^{\mu\nu}$. Write out its covariant version $H_{\mu\nu}$ in matrix form, expressing each element of $H_{\mu\nu}$ in terms of the elements of $H^{\mu\nu}$.

Physics 110B
Homework #5

#1. (a)



$\gamma\beta = 4/3$

$q_2 \cdot q_2 = q_3 \cdot q_3 = 0$

$t = \tau\gamma$

$x = \beta c t = \beta c \gamma \tau = \boxed{4/3\, c\tau = x}$

(b) $p_1 = (\gamma mc,\ \gamma\beta mc, 0, 0)$ $\quad q_2 = (q_0,\ q_0 \cos\psi/2,\ q_0 \sin\psi/2, 0)$

$q_3 = (q_0,\ q_0 \cos\psi/2,\ -q_0 \sin\psi/2, 0)$

Energy Conservation:

$\gamma mc^2 = (q_0 + q_0)c \Rightarrow \underline{\underline{q_0 = \tfrac{1}{2}\gamma mc}}$

Momentum Conservation:

$\gamma\beta mc = 2 q_0 \cos\psi/2 = \gamma mc \cos\psi/2 \Rightarrow \underline{\underline{\cos\psi/2 = \beta}}$

Energy-Momentum Conservation:

$p_1 = q_2 + q_3$

$p_1 \cdot p_1 = \cancel{q_2 \cdot q_2}^{\,0} + 2 q_2 \cdot q_3 + \cancel{q_3 \cdot q_3}^{\,0}$

$m^2c^2 = 2q_0^2(1 - \cos^2\psi/2 + \sin^2\psi/2)$

$m^2c^2 = \tfrac{1}{2}\gamma^2 m^2 c^2 (2\sin^2\psi/2)$

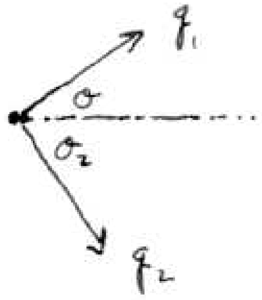
$\underline{\underline{1/\gamma = \sin\psi/2}}$

$$\tan \psi/2 = \frac{\sin \psi/2}{\cos \psi/2} = \frac{1}{\gamma\beta} = \frac{3}{4}$$

$$\psi/2 = \arctan 3/4 = 36.87^\circ$$

$$\Longrightarrow \boxed{\psi = 73.7^\circ}$$

#2. $\underset{\substack{E = \gamma m \\ p_1}}{\overrightarrow{m}} + \underset{p_2}{m} \longrightarrow$



$q_1 \cdot q_1 = q_2 \cdot q_2 = 0$

$$p_1 = (\gamma mc, \gamma\beta mc, 0, 0) \qquad q_1 = (E'/c, \tfrac{E'}{c}\cos\theta, \tfrac{E'}{c}\sin\theta, 0)$$

$$p_2 = (mc, 0, 0, 0) \qquad q_2 = (\tfrac{E_2}{c}, \tfrac{E_2}{c}\cos\theta_2, \tfrac{E_2}{c}\sin\theta_2, 0)$$

$$p_1 + p_2 = q_1 + q_2$$

$$p_1 + p_2 - q_1 = q_2$$

$$p_1 \cdot p_1 + p_2 \cdot p_2 + \cancelto{0}{q_1 \cdot q_1} + 2p_1 \cdot p_2 - 2p_1 \cdot q_1 - 2p_2 \cdot q_1 = \cancelto{0}{q_2 \cdot q_2}$$

$$m^2c^2 + m^2c^2 + 0 + 2\gamma m^2c^2 = 2(\gamma \tfrac{E'}{c} mc - \gamma\beta mc \tfrac{E'}{c}\cos\theta) + 2\tfrac{E'}{c} mc$$

$$\Longrightarrow 2(1+\gamma) m^2c^2 = 2\tfrac{E'}{c} mc (1 + \gamma - \gamma\beta\cos\theta)$$

$$\frac{mc^2}{E'} = 1 - \frac{\gamma\beta}{1+\gamma}\cos\theta$$

$$\frac{mc^2}{E'} = 1 - \frac{\gamma(1 - 1/\gamma^2)^{1/2}}{1+\gamma}\cos\theta$$

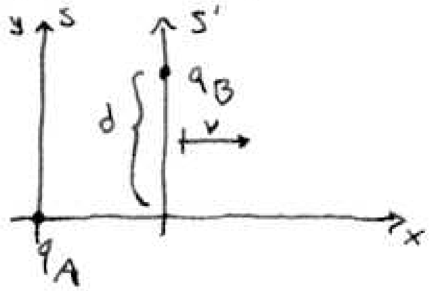
$$= 1 - \frac{\gamma \, 1/\gamma (\gamma^2 - 1)^{1/2}}{1+\gamma}\cos\theta$$

$$= 1 - \frac{\left((\gamma-1)(\gamma+1)\right)^{1/2}}{\gamma+1}\cos\theta$$

$$\Rightarrow \boxed{\frac{mc^2}{E'} = 1 - \sqrt{\frac{\gamma-1}{\gamma+1}}\cos\theta}$$

#3. (Griffiths 12.44)

(a)



Fields of A at B: $\bar{E} = \frac{1}{4\pi\epsilon_0}\frac{q_A}{d^2}\hat{y}$; $\bar{B} = 0$

So, the force on q_B is $\boxed{\bar{F} = \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{d^2}\hat{y}}$

(b) (i.) $F_y = \frac{dp_y}{dt} = \frac{dp_y'}{\gamma dt' + \gamma\beta/c\, dx'} = \frac{dp_y'/dt'}{\gamma\left(1 - \frac{\beta}{c}\frac{dx'}{dt'}\right)}$

$$= \frac{F_y'}{\gamma(1 - \beta \overset{\to 0}{\cancel{u_x}}/c)} = F_y'/\gamma$$

$$\Rightarrow F_y' = \gamma F_y = \boxed{\frac{\gamma}{4\pi\epsilon_0}\frac{q_A q_B}{d^2}\hat{y} = F'}$$

(ii) $E_y' = \gamma(E_y - vB_z)$ (eq 12.108)

$E_y' = \gamma \frac{1}{4\pi\epsilon_0}\frac{q_A}{d^2}\hat{y}$

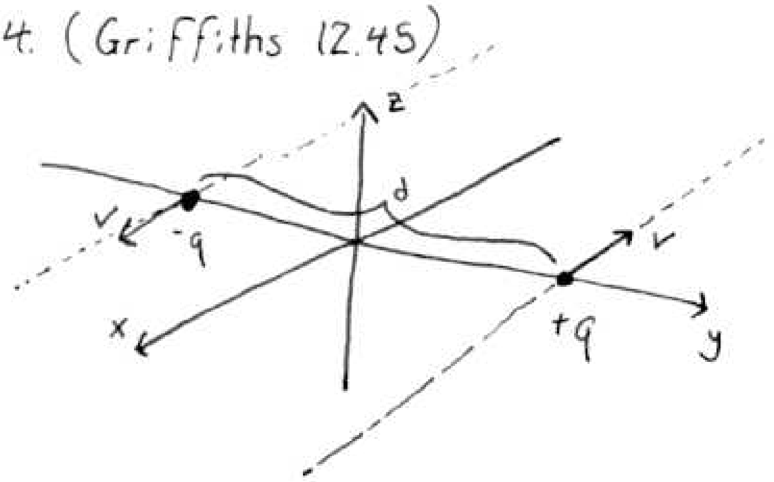
$B_y' = \gamma(B_y + v/c^2 E_z) = 0$

$$E_z' = \gamma(E_z + vB_y) = 0$$

$$B_z' = \gamma(B_z - \tfrac{v}{c^2}E_y) = \gamma \frac{v}{c^2}\frac{1}{4\pi\epsilon_0}\frac{q_A}{d^2}\hat{y}$$

$$\bar{F}' = q_B(\bar{E}' + \cancel{\bar{v}}^{\,0} \times \bar{B}') = \boxed{q_B \frac{\gamma}{4\pi\epsilon_0}\frac{q_A}{d^2}\hat{y} = F'}$$

#4. (Griffiths 12.45)



We will start with System C (-q at rest) and then transform (Lorentz) into the various other systems. The check will be $E^2 - c^2B^2$ since this is a Lorentz invariant.

System C (-q at rest):

$$V_c = 0 \qquad \underline{\underline{\bar{E} = -\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{y}}}\ ; \quad \underline{\underline{\bar{B} = 0}}\ ; \quad \underline{\underline{\bar{F} = \bar{q}\bar{E} = -\frac{q^2}{4\pi\epsilon_0}\frac{1}{r^2}\hat{y}}}$$

Check: $E^2 - c^2B^2 = \dfrac{q^2}{(4\pi\epsilon_0)^2}\dfrac{1}{r^4}$

System B (+q at rest):

$$V_B = \frac{v+v}{1+v^2/c} = \frac{2v}{1+v^2/c^2}$$

$$\gamma_B = \left(1 - \frac{4v^2/c^2}{(1+v^2/c^2)^2}\right)^{1/2} = \frac{1+v^2/c^2}{\left(1 - \frac{2v^2}{c^2} + \frac{v^4}{c^4}\right)^{1/2}}$$

$$= \frac{1 + v^2/c^2}{\left(\left(1 - v^2/c^2\right)^2\right)^{1/2}} = \frac{1 + v^2/c^2}{1 - v^2/c^2} = \gamma^2\left(1 + \frac{v^2}{c^2}\right)$$

$$\gamma \equiv \left(1 - v^2/c^2\right)^{-1/2}$$

Thus, by Lorentz transforming:

$$\bar{E} = -\gamma_B \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{y} = \underline{-\frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \gamma^2\left(1 + \frac{v^2}{c^2}\right) \hat{y}}$$

$$\bar{B} = \gamma_B \frac{v_B}{c^2} E_y = -\gamma^2\left(1 + v^2/c^2\right) \frac{2v/c^2}{1 + v^2/c^2} \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{z}$$

$$= \underline{\underline{-\gamma^2\, 2v/c^2 \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{z}}}$$

Check: $$E^2 - c^2 B^2 = \left(\frac{q}{4\pi\epsilon_0} \frac{\hat{y}}{r^2}\right)^2 \gamma^4\left(1 + \frac{v^2}{c^2}\right)^2 - c^2 \gamma^4 \frac{4v^2}{c^4}\left(\frac{q}{4\pi\epsilon_0} \frac{\hat{z}}{r^2}\right)^2$$

$$= \left(\frac{q}{4\pi\epsilon_0} \frac{1}{r^2}\right)^2 \left(\gamma^4\left(\left(1 + v^2/c^2\right)^2 - 4v^2/c^2\right)\right)$$

$$= \left(\frac{q}{4\pi\epsilon_0} \frac{1}{r^2}\right)^2 \left(\gamma^4\left(1 - 2v^2/c^2 + v^4/c^4\right)\right)$$

$$= \left(\frac{q}{4\pi\epsilon_0} \frac{1}{r^2}\right)^2 \left(\gamma^4\left(1 - v^2/c^2\right)^2\right) = \left(\frac{q}{4\pi\epsilon_0} \frac{1}{r^2}\right)^2$$

$$\bar{F} = q\left(\bar{E} + v \times \bar{B}\right) \qquad v = 0 \text{ for } +q \text{ at rest}$$

$$= q\bar{E} = \underline{\underline{\frac{-q^2}{4\pi\epsilon_0} \frac{1}{r^2} \gamma^2\left(1 + v^2/c^2\right) \hat{y}}}$$

System A (Figure given at the beginning of the problem)

$$V_A = -V\hat{x}$$

$$\gamma_A = \gamma$$

$$\bar{E} = -\gamma_A \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{y} = \underline{\underline{-\frac{q}{4\pi\epsilon_0} \frac{\gamma}{r^2} \hat{y}}}$$

$$\bar{B} = \gamma_A \, v_{A}/c^2 \, E_y = \underline{\underline{-\frac{v}{c^2} \frac{q}{4\pi\epsilon_0} \frac{\gamma}{r^2} \hat{z}}}$$

$$\bar{F} = q(\bar{E} + \bar{v} \times \bar{B}) = -\frac{q^2}{4\pi\epsilon_0} \frac{1}{r^2} \left(\gamma \hat{y} - \gamma v^2/c^2 (\hat{x} \times \hat{z}) \right)$$

$$= \underline{-\frac{q^2}{4\pi\epsilon_0} \frac{\gamma}{r^2} \left(1 + \frac{v^2}{c^2}\right) \hat{y}}$$

Check: $$E^2 - c^2B^2 = \left(-\frac{q}{4\pi\epsilon_0} \frac{\hat{y}}{r^2} \right)^2 \left(\gamma^2 - c^2 \frac{v^2}{c^4} \gamma^2 \right)$$

$$= \left(\frac{-q}{4\pi\epsilon} \frac{1}{r^2} \right)^2 \gamma^2 \left(1 - v^2/c^2 \right) = \left(\frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \right)^2$$

Placing the results together in a table we have:

	System A	System B	System C
E at +q due to -q	$-\frac{q}{4\pi\epsilon_0 r^2} \gamma \hat{y}$	$-\frac{q}{4\pi\epsilon_0 r^2} \gamma^2 \left(1 + \frac{v^2}{c^2}\right) \hat{y}$	$-\frac{q}{4\pi\epsilon_0 r^2} \hat{y}$
B at -q due to -q	$-\frac{q}{4\pi\epsilon_0 r^2} \frac{v}{c^2} \gamma \hat{z}$	$-\frac{q}{4\pi\epsilon_0 r^2} \frac{2v}{c^2} \gamma^2 \hat{z}$	0
F on +q due to -q	$-\frac{q^2}{4\pi\epsilon_0 r^2} \gamma \left(1 + \frac{v^2}{c^2}\right) \hat{y}$	$-\frac{q^2}{4\pi\epsilon_0 r^2} \gamma^2 \left(1 + \frac{v^2}{c^2}\right) \hat{y}$	$-\frac{q}{4\pi\epsilon_0 r^2} \hat{y}$

#5. (Griffiths 12.46)

(a) Using (Eq. 12.108):

$$\bar{E}' \cdot \bar{B}' = E_x' B_x' + E_y' B_y' + E_z' B_z'$$

$$= E_x B_x + \gamma^2 (E_y - vB_z)(B_y + v/c^2 E_z) + \gamma (E_z + vB_y)(B_z - \frac{v}{c^2} E_y)$$

$$= E_x B_x + \gamma^2 \Big(E_y B_y + v/c^2 \cancel{E_y} E_z - vB_{\cancel{y}} B_z + E_z B_z - \frac{v^2}{c^2} E_z B_z$$

$$- \tfrac{v}{c^2} \cancel{E_y} B_z + v \cancel{B_y} B_z - \tfrac{v^2}{c^2} E_y B_y \Big)$$

$$= E_x B_x + \gamma^2 \Big(E_y B_y (1 - \tfrac{v^2}{c^2}) + E_z B_z (1 - \tfrac{v^2}{c^2}) \Big)$$

$$= E_x B_x + E_y B_y + E_z B_z$$

$$\boxed{\bar{E}' \cdot \bar{B}' = \bar{E} \cdot \bar{B}} \quad \text{QED}$$

(b) $$E'^2 - c^2 B'^2 = \Big(E_x^2 + \gamma^2 (E_y - vB_z)^2 + \gamma^2 (E_z + vB_y)^2 \Big)$$

$$- c^2 \Big(B_x^2 + \gamma^2 (B_y + \tfrac{v}{c^2} E_z)^2 + \gamma^2 (B_z - \tfrac{v}{c^2} E_y)^2 \Big)$$

$$= E_x^2 + \gamma^2 \Big(E_y^2 - 2 \cancel{E_y v B_z} + v^2 B_z^2 + E_z^2 + \cancel{2 E_z v B_y} + v^2 B_y^2$$

$$- c^2 B_y^2 - \cancel{c^2 2 \tfrac{v}{c^2} B_y E_z} - c^2 \tfrac{v^2}{c^4} E_z^2 - c^2 B_z^2 + \cancel{c^2 2 \tfrac{v}{c^2} B_z E_y}$$

$$- c^2 \tfrac{v^2}{c^4} E_y^2 \Big) - c^2 B_x^2$$

$$= E_x^2 - c^2 B_x^2 + \gamma^2 \Big(E_y^2 (1 - \tfrac{v^2}{c^2}) + E_z^2 (1 - \tfrac{v^2}{c^2}) - c^2 B_y^2 (1 - \tfrac{v^2}{c^2})$$

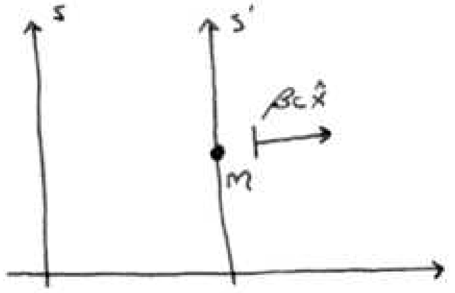
$$- c^2 B_z^2 (1 - \tfrac{v^2}{c^2}) \Big)$$

$$= (E_x^2 + E_y^2 + E_z^2) - c^2 (B_x^2 + B_y^2 + B_z^2)$$

$$\boxed{E'^2 - c^2 B'^2 = E^2 - c^2 B^2} \quad \text{QED}$$

(c) $\boxed{\text{No,}}$ it is not possible to find another system in which the electric field is zero at P. For if $B = 0$ in one system, then $(E^2 - c^2 B^2)$ is positive. Since it is invariant, it must be positive in any system. Therefore $E \neq 0$ in all systems.

#6.



$$p_\mu h^\mu = 0$$

$$p_\mu = (\gamma mc, \gamma\beta mc, 0, 0) \qquad h^\mu = (h_0, h_1, h_2, h_3)$$

$$p'_\mu = (mc, 0, 0, 0) \qquad h'^\mu = (h'_0, h'_1, h'_2, h'_3)$$

$$p'_\mu h'^\mu = mch'_0 = 0 \implies \underline{\underline{h'_0 = 0}}$$

$$p_\mu h^\mu = \gamma mch_0 - \gamma\beta mch_1 = 0 \implies \underline{\underline{h_0 = \beta h_1}}$$

Doing a Lorenz transformation:

$$h'_0 = \gamma(h_0 - \beta h_1) = 0$$

$$h'_1 = \gamma(-\beta h_0 + h_1) = \gamma(-\beta^2 + 1)h_1 = h_1/\gamma$$

$$h'_2 = h_2$$

$$h_3 = h'_3$$

Thus, $\boxed{\begin{array}{ll} h_1 = \gamma h'_1 \;; & h_0 = \gamma\beta h'_1 \\ h_2 = h'_2 \;; & h_3 = h'_3 \end{array}}$

#7. $h_\mu \equiv g_{\mu\nu} h^\nu$; $h_\mu h^\mu$ = invariant length squared

(a) $$\boxed{g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}$$

(b) $T_{\mu\nu} \equiv g_{\mu\alpha} T^{\alpha\beta} g_{\beta\nu}$

Using the above transformation:

$$g_{\mu\nu} \equiv g_{\mu\alpha}\, g^{\alpha\beta}\, g_{\beta\nu}$$

$$① \quad g_{\mu\alpha}^{-1}\, g_{\mu\nu}\, g_{\beta\nu}^{-1} = g_{\mu\alpha}^{-1}\, g_{\mu\alpha}\, g^{\alpha\beta}\, g_{\beta\nu}\, g_{\beta\nu}^{-1}$$

$$g_{\mu\alpha}^{-1} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = g_{\mu\alpha}$$

Thus eq ① becomes:

$$g^{\alpha\beta} = g_{\mu\alpha}^{-1}\, g_{\mu\nu}\, g_{\beta\nu}^{-1} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}\begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}\begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = g_{\alpha\beta}$$

Thus, $\boxed{g^{\mu\nu} = g_{\mu\nu}}$

(c) We begin with the result from part (a)

$$g_{\mu\alpha} = g^{\mu\alpha} \implies g_{\mu\alpha}(g^{\nu\alpha})^{-1} = g^{\mu\alpha}(g^{\nu\alpha})^{-1}$$

$$\Rightarrow g_{\mu\alpha}\, g^{\alpha\nu} = \underbrace{g^{\mu\alpha}(g^{\nu\alpha})^{-1}}_{\delta_\mu^\nu}$$

Thus, $\boxed{g_{\mu\alpha}\, g^{\alpha\nu} = \delta_\mu^\nu}$

#8. $H^{\mu\nu}$ is antisymmetric w/ 6 independent components.

$$H^{\mu\nu} = \begin{pmatrix} 0 & H^{01} & H^{02} & H^{03} \\ H^{10} & 0 & H^{12} & H^{13} \\ H^{20} & H^{21} & 0 & H^{23} \\ H^{30} & H^{31} & H^{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & H^{01} & H^{02} & H^{03} \\ -H^{01} & 0 & H^{12} & H^{13} \\ -H^{02} & -H^{12} & 0 & H^{23} \\ -H^{03} & -H^{13} & -H^{23} & 0 \end{pmatrix}$$

Using: $H_{\mu\nu} = g_{\mu\alpha} H^{\alpha\beta} g_{\beta\nu}$

$$= \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} H^{\alpha\beta} \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} 0 & -H^{01} & -H^{02} & -H^{03} \\ -H^{01} & 0 & -H^{12} & -H^{13} \\ -H^{02} & H^{12} & 0 & H^{23} \\ -H^{03} & H^{13} & H^{23} & 0 \end{pmatrix}$$

$$\boxed{H_{\mu\nu} = \begin{pmatrix} 0 & -H^{01} & -H^{02} & -H^{03} \\ H^{01} & 0 & H^{12} & H^{13} \\ H^{02} & -H^{12} & 0 & H^{23} \\ H^{03} & -H^{12} & -H^{23} & 0 \end{pmatrix}}$$

University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

Problem Set 6

1. *Relativistic transformation of a particle's polar angle.* Consider the usual Lorentz frames $\mathcal{S}$ and $\mathcal{S}'$, with spatial origins coincident at $t = t' = 0$. As usual, frame $\mathcal{S}'$ moves in the $\hat{x}$ or $\hat{x}'$ direction with velocity βc with respect to frame $\mathcal{S}$. A particle is emitted by a radioactive source that is at rest with respect to $\mathcal{S}'$. As seen by an observer in $\mathcal{S}'$, the particle travels with velocity $\beta' c$ at an angle θ' with respect to the $\hat{x}'$ direction. However, as seen by an observer who is at rest with respect to the frame $\mathcal{S}$, prove that the particle makes a different angle θ with respect to the $\hat{x}$ direction, where

$$\tan\theta = \frac{\sin\theta'}{\gamma\big(\cos\theta' + (\beta/\beta')\big)} \ .$$

2. and **3.** (double credit problem)
Violation of time-reversal invariance was discovered in 1964 in the weak decay

$$K_L^0 \to \pi^+\pi^- \ ,$$

where the K_L^0 and $\pi^\pm$ are quark-antiquark pairs (including a strange quark in the K_L^0 case); a kaon has $\approx \frac{7}{2}$ of a pion's mass. In its own rest frame, the (spin 0) kaon decays isotropically. Suppose that the kaons compose a finite beam whose momentum per particle is $2m_K c$ ($\approx$ 1 GeV/c). With respect to the beam direction, find the laboratory angle θ at which the flux of decay pions per unit solid angle, $dN/d\Omega dt$, is *infinite*. [*Hint*: the answer is *not* $\theta = 90°$.]

4. Define the contravariant four-vectors

$$\begin{aligned} A^\mu &\equiv \{V/c, \mathbf{A}\} \\ J^\mu &\equiv \{c\rho, \mathbf{J}\} \\ p^\mu &\equiv \{E/c, \mathbf{p}\} \\ k^\mu &\equiv \{\omega/c, \mathbf{k}\} \\ \partial^\mu &\equiv \{\partial/c\partial t, -\nabla\} \ . \end{aligned}$$

Use the convention that repeated Greek indices are summed from 0 to 3. Employing primarily contravariant four-vectors, but making use of covariant four-vectors where appropriate, write a manifestly Lorentz invariant equation that is equivalent to
(**a**) the generalized de Broglie relation.
(**b**) conservation of electric charge.
(**c**) the Lorentz gauge condition.
(**d**) the wave equation, including sources, for the electromagnetic potentials in Lorentz gauge.

5. An object a^μ is a (contravariant) four-vector if it transforms (between frames as defined in Problem 1) according to

$$a'^\mu = \Lambda^\mu_\nu a^\nu \ ,$$

where Λ is the (symmetric) 4×4 Lorentz transformation matrix. (Conventionally, the superscript labels the row and the subscript labels the column, but this makes no difference for a symmetric matrix.) Covariant four-vectors instead transform according to

$$a'_\mu = (\Lambda^{-1})^\nu_\mu a_\nu$$

(otherwise the scalar product $a_\mu a^\mu = a'_\mu a'^\mu$ would not remain invariant for different Lorentz frames). Consider now an (arbitrary) four-tensor $H^{\mu\nu}$. In frame $\mathcal{S}$, $H^{\mu\nu}$ contracts with covariant four-vector a_ν to yield contravariant four-vector b^μ, according to

$$b^\mu = H^{\mu\nu} a_\nu \ .$$

In the frame $\mathcal{S}'$, requiring $H^{\mu\nu}$ to satisfy the transformation properties of a four-tensor, we define $H'^{\mu\nu}$ so that

$$b'^\mu = H'^{\mu\nu} a'_\nu \ .$$

Prove that

$$H'^{\mu\nu} = \Lambda^\mu_\rho H^{\rho\sigma} \Lambda^\nu_\sigma \ .$$

This defines the Lorentz transformation property of a four-tensor.

6. Consider the antisymmetric *electromagnetic field strength tensor*

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \ .$$

Prove that $F^{\mu\nu}$ is a four-tensor, *i.e.* it transforms according to the results of Problem 5.

7. Using the definitions of ∂^μ and A^μ, show by explicit calculation, element by element, that the covariant electromagnetic field strength tensor is equal to

$$F = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix} .$$

(The sign of this result is opposite to that of Griffiths; this is expected from his use of a metric tensor with sign opposite to the standard.)

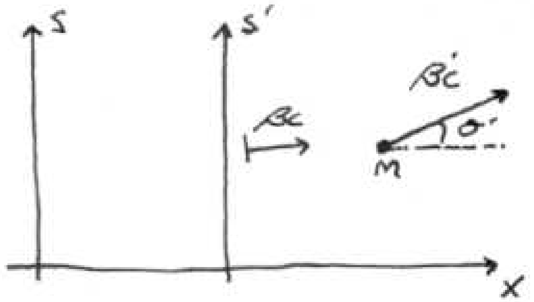
8. Prove that the equation

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

is equivalent (in vacuum) to the two Maxwell equations which involve sources. (The two source-free Maxwell equations are already required to be true by the definition of A^μ.)

Physics 110B
Homework #6

#1.



$$\gamma' \equiv (1-\beta'^2)^{-1/2}$$

$$p' = (mc\gamma';\ m\beta'c\gamma'\cos\theta',\ m\beta'c\gamma'\sin\theta';\ 0)$$
$$= (p_0',\ p_1',\ p_2',\ p_3')$$
$$p = (p_0,\ p\cos\theta,\ p\sin\theta,\ 0) = (p_0,\ p_1,\ p_2,\ p_3)$$

Now, using a Lorentz transformation:

$$p_0 = \gamma(p_0' + \beta p_1') \quad ; \quad p_2 = p_2'$$
$$p_1 = \gamma(p_1' + \beta p_0') \quad ; \quad p_3 = p_3'$$

So,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{p\sin\theta}{p\cos\theta} = \frac{p_2}{p_1} = \frac{p_2'}{\gamma(p_1' + \beta p_0')}$$
$$= \frac{\gamma' m\beta' c\sin\theta'}{\gamma(m\beta'c\gamma'\cos\theta' + \beta mc\gamma')}$$

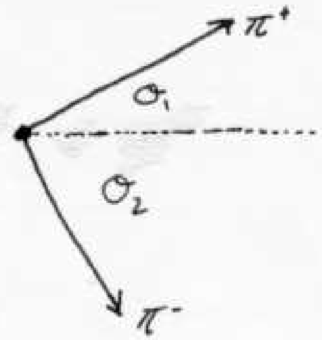
$$\boxed{\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta/\beta')}}$$

#2. and #3.

$\overset{\longmapsto}{p = 2 m_K c}$

$K_L^\circ \longrightarrow$

$m_K = 7/2\, m_\pi$



In the rest frame the Kaon decays isotropical, therefore the flux of decay pions per unit solid angle is constant:

$$\frac{dN}{d\Omega' dt} = \text{Constant} \qquad (\text{Since, isotropic})$$

In the lab frame the flux of decay pions per unit solid angle is:

$$\frac{dN}{d\Omega\, dt} = \underbrace{\frac{dN}{d\Omega' dt}}_{\text{const.}} \cdot \frac{d\Omega'}{d\Omega} = \text{const.}\ \frac{\sin\theta'\, d\theta'\, d\varphi'}{\sin\theta\, d\theta\, d\varphi}$$

Since φ is perpendicular to the direction of motion it remains the same both in the Lab and Rest Frames: $\varphi = \varphi'$

Thus, $$\frac{dN}{d\Omega\, dt} = \text{const.}\ \frac{\sin\theta'}{\sin\theta}\ \frac{d\theta'}{d\theta}$$

When $\sin\theta = 0$ $(\theta = 0, \pi)$ the flux per solid angle is infinite, however, these are the trivial solutions. The flux per solid angle will also go to infinity when $d\theta'/d\theta \rightarrow \infty$:

$$\frac{dN}{d\Omega dt} \longrightarrow \infty \quad \text{when} \quad \frac{d\theta'}{d\theta} \longrightarrow \infty$$

$$\Longrightarrow \quad \text{Or when} \quad \frac{d\theta}{d\theta'} \rightarrow 0$$

So, we need to find when $\frac{d\theta}{d\theta'} = 0$

$$\theta = \arctan\left(\frac{\sin\theta'}{\gamma(\cos\theta' + (\beta/\beta'))}\right) \quad (\text{from problem \#1})$$

$$\frac{d\theta}{d\theta'} = \frac{d}{d\theta'}\left(\arctan x\right) = \frac{dx}{d\theta'}\,\frac{d}{dx}\left(\arctan x\right) = 0$$

$$\Rightarrow \frac{dx}{d\theta'} = 0$$

$$\Rightarrow \frac{d}{d\theta'}\left(\frac{\sin\theta'}{\gamma(\cos\theta' + \beta/\beta')}\right) = \frac{\cos\theta'(\cos\theta' + \frac{\beta}{\beta'}) - \sin\theta'(-\sin\theta')}{\gamma(\cos\theta' + \beta/\beta')^2} = 0$$

$$\Rightarrow \cos\theta'(\cos\theta' + \beta/\beta') + \sin^2\theta' = 0$$

$$\cos^2\theta' + \sin^2\theta' + (\cos\theta')\beta/\beta' = 0$$

$$(\cos\theta')\beta/\beta' = -1$$

$$\boxed{\arccos\left(-\beta'/\beta\right) = \theta'}$$

Thus, we need to find β and β':

$$p_{K^0_L} = (E_K/c,\ 2m_K c,\ 0,\ 0)$$

$$p_K \cdot p_K = E_K^2 - 4m_K^2 c^2 = M_K^2 c^2 \quad \Rightarrow \quad E_K = \sqrt{5}\, m_K c$$

Since, $p_K = (\frac{E_K}{c}, \bar{p}_K) = (\gamma m_K c, \gamma\bar{\beta} c m_K)$

Thus, $\frac{E_K/c}{p_K} = \frac{\gamma m_K c}{\gamma\beta c m_K} = \frac{1}{\beta}$

So,

$$\beta = \frac{p_K}{E_K} = \frac{2m_K c}{\sqrt{5}\, m_K c} = \boxed{\frac{2}{\sqrt{5}} = \beta}$$

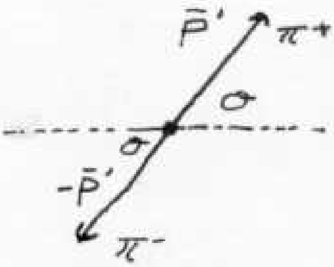
$$p_K = \gamma\beta c m_K = 2m_K c$$

$$\gamma\beta = 2 \quad \Rightarrow \quad \boxed{\gamma = \sqrt{5}}$$

In the Rest Frame,

m_K $\longrightarrow$

$E'_K = m_K c^2$



From energy conservation:

$$E'_K = E'_{\pi^+} + E'_{\pi^-}$$

$$m_K c^2 = \sqrt{c^2 p'^2 + m_\pi^2 c^4} + \sqrt{p'^2 c^2 + m_\pi^2 c^4}$$

$$\left(\frac{7}{4} m_\pi c^2\right)^2 = c^2 p'^2 + m_\pi^2 c^4$$

$$\frac{33}{4\cdot 4} m_\pi^2 = p'^2/c^2 \implies p'_\pi = \frac{\sqrt{33}}{4} m_\pi c$$

$$E'_\pi = E'_K/2 = m_K c^2/2 = \tfrac{7}{4} m_\pi c^2$$

Thus, $\implies \beta' = \dfrac{p'_\pi c}{E'_\pi} = \dfrac{\sqrt{33}/4 \ m_\pi c^2}{7/4 \ m_\pi c^2} = \boxed{\dfrac{\sqrt{33}}{7} = \beta'}$

$$\implies \sigma' = \arccos\left(-\frac{\beta'}{\beta}\right) = \arccos\left(\left(-\frac{2}{\sqrt{5}}\frac{7}{\sqrt{33}}\right)^{-1}\right) = 156°$$

$$\implies \theta = \arctan\left(\frac{\sin\theta'}{\gamma(\cos\theta' + \beta/\beta')}\right) = \arctan\left(\frac{\sin 156}{\sqrt{5}\left(\cos 156 + \frac{2\cdot 7}{\sqrt{5\cdot 33}}\right)}\right)$$

$$\boxed{\theta = 45.9°}$$

the flux per unit solid angle goes to infinity at this angle

#4. $A^\mu \equiv \{ V/c, \bar{A} \}$, $k^\mu \equiv \{ \omega/c, \bar{k} \}$

$J^\mu \equiv \{ c\rho, \bar{J} \}$, $\partial^\mu \equiv \{ \partial/c\partial t, -\bar{\nabla} \}$

$p^\mu \equiv \{ E/c, \bar{p} \}$

(a) Generalized de Broglie relation:

$$p = h/\lambda = 2\pi\hbar/\lambda = \hbar k \quad , \quad E = \hbar\omega$$

$$\Rightarrow \left(\frac{E}{c}, \bar{p} \right) = \left(\omega/c, \bar{k} \right)\hbar \Longrightarrow \boxed{p^\mu = \hbar k^\mu}$$

(b) Conservation of electric charge:

$$\frac{\partial \rho}{\partial t} + \bar{\nabla}\cdot\bar{J} = 0 \Rightarrow \left(\frac{\partial}{c\partial t}, \bar{\nabla} \right)\cdot\left(c\rho, -J \right) = 0$$

$$\Rightarrow \boxed{\partial_\mu J^\mu = 0}$$

(c) Lorentz Gauge condition:

$$\bar{\nabla}\cdot\bar{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0 \Rightarrow \left(\frac{\partial}{c\partial t}, \bar{\nabla} \right)\cdot\left(\frac{V}{c}, -\bar{A} \right) = 0$$

$$\Rightarrow \boxed{\partial_\mu A^\mu = 0}$$

(d) the Wave Equation, including sources, for the electromagnetic potentials:

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} V = -\rho/\epsilon_0 \quad , \quad \nabla^2 \bar{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\bar{A} = -\mu_0 \bar{J}$$

$$\Rightarrow \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \Rightarrow \partial_\mu \partial^\mu (V/c, \bar{A}) = \left(\frac{\rho}{\epsilon_0}, \mu_0 \bar{J} \right)$$

$$\Rightarrow \partial_\mu \partial^\mu A^\nu = \mu_0 \left(\frac{\rho}{\epsilon_0 \mu_0}, \bar{J} \right) \Rightarrow \boxed{\partial_\mu \partial^\mu A^\nu = \mu_0 J^\nu}$$

#5. • $a'^{\mu} = \Lambda^{\mu}_{\ \nu}\, a^{\nu}$ (contravariant transformation)

• $a'_{\mu} = (\Lambda^{-1})^{\nu}_{\ \mu}\, a_{\nu}$ (covariant transformation)

We start with:

① $b^{\rho} = H^{\rho\sigma} a_{\sigma}$

and

② $b'^{\mu} = H'^{\mu\nu} a'_{\nu} \Rightarrow \Lambda^{\mu}_{\ \rho} b^{\rho} = H'^{\mu\nu} (\Lambda^{-1})^{\sigma}_{\ \nu} a_{\sigma}$

$\Rightarrow b^{\rho} = (\Lambda^{\mu}_{\ \rho})^{-1} H'^{\mu\nu} (\Lambda^{-1})^{\sigma}_{\ \nu} a_{\sigma}$

(comparing with eq ①) $\Rightarrow H^{\rho\sigma} = (\Lambda^{\mu}_{\ \rho})^{-1} H'^{\mu\nu} (\Lambda^{-1})^{\sigma}_{\ \nu}$

$H'^{\mu\nu} = \Lambda^{\mu}_{\ \rho} H^{\rho\sigma} \left((\Lambda^{-1})^{\sigma}_{\ \nu}\right)^{-1}$

$\Rightarrow \boxed{H'^{\mu\nu} = \Lambda^{\mu}_{\ \rho} H^{\rho\sigma} \Lambda^{\nu}_{\ \sigma}}$ QED

#6. Prove: $F'^{\mu\nu} = \Lambda^{\mu}_{\ \rho} F^{\rho\sigma} \Lambda^{\nu}_{\ \sigma}$ ①

where, $F^{\mu\nu} \equiv \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$

(Starting with the LHS of eq ①) $\Rightarrow F'^{\mu\nu} = \partial'^{\mu} A'^{\nu} - \partial'^{\nu} A'^{\mu}$

(Using transformations from problem #5) $= (\Lambda^{\mu}_{\ \sigma} \partial^{\sigma})(\Lambda^{\nu}_{\ \rho} A^{\rho}) - (\Lambda^{\nu}_{\ \alpha} \partial^{\alpha})(\Lambda^{\mu}_{\ \beta} A^{\beta})$

(Relabeling dummy indices) $= (\Lambda^{\mu}_{\ \rho} \partial^{\rho})(A^{\sigma} \Lambda^{\nu}_{\ \sigma}) - (\Lambda^{\nu}_{\ \sigma} \partial^{\sigma})(A^{\rho} \Lambda^{\mu}_{\ \rho})$

$= \Lambda^{\mu}_{\ \rho} (\partial^{\rho} A^{\sigma} - \partial^{\sigma} A^{\rho}) \Lambda^{\nu}_{\ \sigma}$

$\Rightarrow \boxed{F'^{\mu\nu} = \Lambda^{\mu}_{\ \rho} F^{\rho\sigma} \Lambda^{\nu}_{\ \sigma}}$ QED

#7. $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

- $F^{00} = F^{11} = F^{22} = F^{33} = 0 \xRightarrow{\text{since}} \boxed{F^{\alpha\alpha} = \partial^\alpha A^\alpha - \partial^\alpha A^\alpha = 0}$

- $F^{0i} = \partial^0 A^i - \partial^i A^0 \quad (i = 1, 2, 3)$

$$= \frac{\partial A^i}{c\partial t} - \frac{1}{c}\frac{\partial V}{\partial x^i}$$

So, $\boxed{F^{0i} = E^i/c} \Rightarrow F^{01} = \frac{E^1}{c} = \frac{E_x}{c}$, $F^{02} = \frac{E^2}{c} = \frac{E_y}{c}$, $F^{03} = \frac{E^3}{c} = \frac{E_z}{c}$

- $F^{i0} = \partial^i A^0 - \partial^0 A^i = -(\partial^0 A^i - \partial^i A^0) = -F^{0i}$

So, $\boxed{F^{i0} = -E^i/c} \Rightarrow F^{10} = -E^1/c$, $F^{20} = -E^2/c$, $F^{30} = -E^3/c$

- $F^{ij} = \partial^i A^j - \partial^j A^i \quad (i, j = 1, 2, 3)$

$$= \epsilon_{ijk}\partial^i A^j$$

$$\boxed{\begin{array}{l} F^{ij} = B^k \\ \text{and} \\ F^{ji} = -B^k \end{array}} \Longrightarrow \begin{array}{lll} F^{12} = B^3, & F^{31} = B^2, & F^{23} = B^1 \\ F^{21} = -B^3, & F^{13} = -B^2, & F^{32} = -B^1 \end{array}$$

Thus,
$$\boxed{F = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix}}$$

#8. Prove: $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu$$

(using the result from problem #4 (d)) $= \mu_0 J^\nu - \partial^\nu \overset{0}{\cancel{\partial_\mu A^\mu}}$ (Lorentz Gauge condition problem #4 part (c))

$$\Longrightarrow \boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu} \quad \text{QED}$$

Continued ...

Above we proved that the statement, $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$, is true. Now, let's show that this statement is equivalent to Maxwell's two source equations:

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu$$

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \mu_0 J^\nu$$

For $\nu = 0$:

$$\Rightarrow \partial_\mu \partial^\mu A^0 - \partial^0 \partial_\mu A^\mu = \mu_0 J^0$$

$$\left(\frac{1}{c^2}\cancel{\frac{\partial^2}{\partial t^2}} - \nabla^2\right)\frac{V}{c} - \frac{\partial}{c\partial t}\left(\bar{\nabla}\cdot\bar{A} + \frac{1}{c^2}\cancel{\frac{\partial V}{\partial t}}\right) = \mu_0 c\rho$$

$$\nabla^2 V + \bar{\nabla}\cdot\frac{\partial\bar{A}}{\partial t} = -\mu_0 c^2 \rho$$

$$\bar{\nabla}\cdot\left(\bar{\nabla}V + \frac{\partial\bar{A}}{\partial t}\right) = -\mu_0 \frac{1}{\mu_0\epsilon_0}\rho$$

$$\boxed{\bar{\nabla}\cdot\bar{E} = \rho/\epsilon_0}$$ Maxwell's source equation

For $\nu = 1, 2, 3 = i$:

$$\Rightarrow \partial_\mu \partial^\mu A^i - \partial^i \partial_\mu A^\mu = \mu_0 J^i$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\bar{A} - (-\bar{\nabla})\left(\bar{\nabla}\cdot\bar{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = \mu_0\bar{J}$$

$$\left(\bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \nabla^2\bar{A}\right) + \frac{1}{c^2}\frac{\partial^2\bar{A}}{\partial t^2} + \frac{1}{c^2}\frac{\partial\bar{\nabla}V}{\partial t} = \mu_0\bar{J}$$

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{A}) + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial\bar{A}}{\partial t} + \bar{\nabla}V\right) = \mu_0\bar{J}$$

$$\boxed{\bar{\nabla}\times\bar{B} = \mu_0 J + \mu_0\epsilon_0\frac{\partial E}{\partial t}}$$ Maxwell's source equation

University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

Problem Set 7

1. Starting from the elements of $F^{\mu\nu}$ as given in PS 6 Problem 7, apply the metric tensor (twice) to find the elements of $F_{\mu\nu}$.

2. Define the contravariant field-strength tensor F and the contravariant *dual* field-strength tensor $\mathcal{F}$ by

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$
$$\mathcal{F}^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \ ,$$

where $\epsilon^{\mu\nu\rho\sigma} = 1$ for $\mu\nu\rho\sigma = 0123$ or any permutation of 0123 that is achieved by an *even* number of interchanges of adjacent indices; $\epsilon^{\mu\nu\rho\sigma} = -1$ for $\mu\nu\rho\sigma = 1023$ or any other permutation of 0123 that is achieved by an *odd* number of interchanges of adjacent indices; and $\epsilon^{\mu\nu\rho\sigma} = 0$ otherwise. By explicit calculation, show that the elements of $\mathcal{F}^{\mu\nu}$ can be obtained from those of $F^{\mu\nu}$ by changing $\mathbf{E}$ into $c\mathbf{B}$ and $c\mathbf{B}$ into $-\mathbf{E}$.

3. By explicit evaluation, show that $\mathcal{F}^{\mu\nu}F_{\mu\nu}$ is proportional to $\mathbf{E}\cdot\mathbf{B}$, and find the constant of proportionality. (Because $\mathcal{F}^{\mu\nu}F_{\mu\nu}$ is obviously a Lorentz scalar, the Lorentz invariance of $\mathbf{E}\cdot\mathbf{B}$ is therefore said to be *manifest*.)

4. The two source-free Maxwell equations are equivalent to the single manifestly Lorentz-invariant equation

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0 \ .$$

Without making any reference to Maxwell's equations, using only formal manipulation, show that $\mathcal{F}$ has this property (*i.e.* its four-divergence vanishes). [*Hint*: Write $\mathcal{F}^{\mu\nu}$ in terms of $\epsilon^{\mu\nu\rho\sigma}$ and $F_{\rho\sigma}$. Then write $F_{\rho\sigma}$ in terms of $\partial_{\rho,\sigma}$ and $A_{\rho,\sigma}$. Finally, make use (twice) of the behavior of $\epsilon^{\mu\nu\rho\sigma}$ under interchange of adjacent indices.]

5. Is it possible to have an electromagnetic field that appears as a purely electric field in one inertial frame and a purely magnetic field in the other? What criteria must (uniform nonzero) $\mathbf{E}$ and $\mathbf{B}$ satisfy such that there exists an inertial frame in which the electromagnetic field is purely magnetic?

6. An infinitely long straight wire of negligible cross-sectional area moves in the $\hat{x}$ direction (parallel to its length) with speed βc relative to the lab. As observed in its *rest* frame, the wire carries a uniform linear charge density λ Coulombs/meter; in that frame, those charges are at rest. In the lab, write the elements of the field strength tensor at the point $(0, y, 0)$.

7. Consider a relativistic particle of mass m and charge e that accelerates in a uniform, static electric field with magnitude E (there is no magnetic field). At $t = 0$ the particle is at rest. Solve for $\eta(t > 0)$, where $\eta \equiv \tanh^{-1}(\beta)$ is the particle's boost parameter or "rapidity".

8. Consider a relativistic particle of mass m and charge e that is in helical motion under the influence of a constant magnetic field of magnitude B (there is no electric field). Its momentum component in the direction of the magnetic field is p_0. Show that the cyclotron angular frequency of this particle is

$$\Omega = \frac{eB}{\gamma_\perp m_{\rm eff}} \ ,$$

where

$$\gamma_\perp \equiv \frac{1}{\sqrt{1-\beta_\perp^2}}$$
$$m_{\rm eff} \equiv \sqrt{m^2 + p_0^2/c^2} \ ,$$

and $c\beta_\perp$ is the component of the particle's velocity that is perpendicular to the magnetic field. (That is, the transverse motion of a particle that moves in a helix is the same as that of a heavier particle that moves purely in a circle.)

Physics 110B
HW#7 Solutions

#1.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix}$$

$$F_{\mu\nu} = g_{\mu\alpha} F^{\alpha\beta} g_{\beta\nu}$$

$$F_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & 0 \\ & 0 & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & 0 \\ & 0 & -1 & \\ & & & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & -1 & & 0 \\ & 0 & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 0 & E^1/c & E^2/c & E^3/c \\ E^1/c & 0 & -B^3 & -B^2 \\ E^2/c & -B^3 & 0 & B^1 \\ E^3/c & B^2 & -B^1 & 0 \end{pmatrix}$$

$$\boxed{F_{\mu\nu} = \begin{pmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & B^3 & B^2 \\ -E^2/c & B^3 & 0 & -B^1 \\ -E^3/c & -B^2 & B^1 & 0 \end{pmatrix}}$$

#2. $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$

$$\mathcal{F}^{\mu\nu} \equiv \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$\mathcal{F}^{\nu\mu} = \tfrac{1}{2} \epsilon^{\nu\mu\rho\sigma} F_{\rho\sigma} = -\tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = -\mathcal{F}^{\mu\nu}$$

Thus, we see that $\mathcal{F}^{\mu\nu}$ is an antisymmetric tensor with:

$$\mathcal{F}^{00} = \mathcal{F}^{11} = \mathcal{F}^{22} = \mathcal{F}^{33} = 0$$

$$\mathcal{F}^{01} = \tfrac{1}{2}\epsilon^{01\rho\sigma}F_{\rho\sigma} = \tfrac{1}{2}\epsilon^{0123}F_{23} + \tfrac{1}{2}\epsilon^{0132}F_{32}$$

$$= \tfrac{1}{2}F_{23} - \tfrac{1}{2}F_{32} = -B^1$$

$$\mathcal{F}^{02} = \tfrac{1}{2}\epsilon^{0213}F_{13} + \tfrac{1}{2}\epsilon^{0231}F_{31} = -\tfrac{1}{2}F_{13} + \tfrac{1}{2}F_{31}$$

$$= -\tfrac{1}{2}B_2 + \tfrac{1}{2}(-B_2) = -B_2$$

$$\mathcal{F}^{03} = \tfrac{1}{2}\epsilon^{0312}F_{12} + \tfrac{1}{2}\epsilon^{0321}F_{21} = \tfrac{1}{2}F_{12} - \tfrac{1}{2}F_{21}$$

$$= -B^3$$

$$\mathcal{F}^{12} = \tfrac{1}{2}\epsilon^{1230}F_{30} + \tfrac{1}{2}\epsilon^{1203}F_{03} = -\tfrac{1}{2}F_{30} + \tfrac{1}{2}F_{03}$$

$$= E^3/c$$

$$\mathcal{F}^{13} = \tfrac{1}{2}\epsilon^{1320}F_{20} + \tfrac{1}{2}\epsilon^{1302}F_{02} = \tfrac{1}{2}F_{20} - \tfrac{1}{2}F_{02}$$

$$= -E^2/c$$

$$\mathcal{F}^{23} = \tfrac{1}{2}\epsilon^{2310}F_{10} + \tfrac{1}{2}\epsilon^{2301}F_{01} = -\tfrac{1}{2}F_{10} + \tfrac{1}{2}F_{01}$$

$$= E^1/c$$

With the above results and the antisymmetric condition, $\mathcal{F}^{\mu\nu} = \mathcal{F}^{\nu\mu}$, we get:

$$\boxed{\mathcal{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3/c & -E^2/c \\ B^2 & -E^3/c & 0 & E^1/c \\ B^3 & E^2/c & -E^1/c & 0 \end{pmatrix}}$$

Thus, we see that $\mathcal{F}^{\mu\nu}$ is obtained from $F^{\mu\nu}$ by changing: $E^i \to cB^i$ and $cB^i \to -E^i$

#3. $\mathcal{F}^{\mu\nu} F_{\mu\nu} = \sum_\mu \sum_\nu \mathcal{F}^{\mu\nu} F_{\mu\nu} = \mathcal{F}^{00}F_{00} + \mathcal{F}^{10}F_{10} + \mathcal{F}^{20}F_{20} + \dots$

$= 0 + \frac{1}{c}(-E^1B^1 - E^2B^2 - E^3B^3 - E^1B^1 + 0 - E^3B^3$

$-E^2B^2 - E^3B^3 + 0 - E^1B^1 - E^3B^3 - E^2B^2 - E^2B^2) + 0$

$= -\frac{4}{c}(E^1_xB^1_x + E^2_yB^2_y + E^3B^3)$

$\boxed{\mathcal{F}^{\mu\nu} F_{\mu\nu} = -4/c\ \bar{E}\cdot\bar{B}}$ $\boxed{\text{the constant of proportionality} = \frac{-4}{c}}$

#4. Prove: $\partial_\mu \mathcal{F}^{\mu\nu} = 0$

$\partial_\mu \mathcal{F}^{\mu\nu} = \partial_\mu \left(\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}\right)$

$= \partial_\mu \left(\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)\right)$

$= \frac{1}{2}\partial_\mu \epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma - \frac{1}{2}\partial_\mu\epsilon^{\mu\nu\rho\sigma}\partial_\sigma A_\rho$

relabeling dummy indices $= \frac{1}{2}\partial_\mu \epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma - \frac{1}{2}\partial_\mu\epsilon^{\mu\nu\sigma\rho}\partial_\rho A_\sigma$

permuting indices in epsilon $= \frac{1}{2}\partial_\mu \epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma + \frac{1}{2}\partial_\mu\epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma$

$= \partial_\mu \epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma$

① $= \epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\sigma$ Now, $\partial_\mu\partial_\rho = \partial_\rho\partial_\mu$

relabeling dummy indices $= \epsilon^{\rho\nu\mu\sigma}\partial_\rho\partial_\mu A_\sigma$

permuting indices in epsilon ② $= -\epsilon^{\mu\nu\rho\sigma}\partial_\rho\partial_\mu A_\sigma = -\epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\sigma$

However, from lines ① and ② we have the impossible statement: $\epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\sigma = -\epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\sigma$

Thus, it follows that,

$$\epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\sigma = 0$$

And, thus:

$$\boxed{\partial_\mu \mathcal{F}^{\mu\nu} = 0}$$

#5. (a) In problem 3 we found the Lorentz invariant quantity:

① $$\mathcal{F}^{\mu\nu}F_{\mu\nu} = -\frac{4}{c}\bar{E}\cdot\bar{B}$$

In a previous problem set we also found the Lorentz invariant quantity:

② $$E^2 - c^2B^2 = \text{constant}$$

Thus, is it possible to have a frame of reference where:

$\bar{E} = \text{constant} = A_1,\ \bar{B} = 0$ while in another frame $\bar{E}' = 0,\ \bar{B} = \text{constant} = A_2$

From eq ② we have,

$$E^2 - c^2B^2 = E'^2 - c^2B^2$$

$$\Rightarrow \quad A_1^2 = -c^2A_2^2 \leftarrow \text{not true}$$

The above statement is not true thus, $\boxed{\text{No, it is not possible to have a purely electric field in one frame, while in another frame a purely Magnetic field}}$

(b) To have the following situation:

$$\bar{E} = A_1,\ \bar{B} = A_2\ ;\quad \bar{E}' = 0,\ \bar{B}' = A_3$$

We will need to satisfy the invariant equations ① and ②:

from eq②: $E^2 - c^2B^2 = E'^2 - c^2B'^2 = -c^2B'^2$

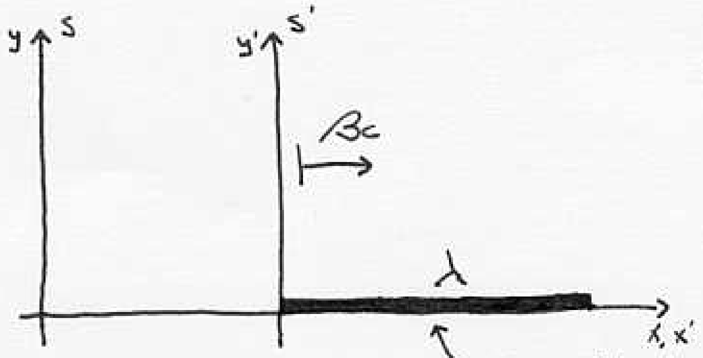
Thus, $E^2 - c^2B^2 < 0$

$\Rightarrow \boxed{E < cB}$ is one condition that must be satisfied

From eq①: $\bar{E}\cdot\bar{B} = \cancelto{0}{\bar{E}'}\cdot\bar{B}' = 0$

Thus, $\Rightarrow \boxed{\bar{E} \perp \bar{B}}$ and another is that the fields have to be perpendicular.

#6.



wire with line charge λ in its rest frame

In the rest frame we use Gauss' Law to calculate the electric field at $(0, y', 0)$

$$\oint \bar{E}'\cdot d\bar{A}' = \frac{Q_{enclosed}}{\epsilon_0} \Rightarrow E' 2\pi r' \ell' = \frac{\lambda \ell'}{\epsilon_0}$$

$$\Rightarrow E' = \frac{\lambda}{2\pi\epsilon_0 r'}\hat{r}' \Rightarrow \bar{E}' = \frac{\lambda}{2\pi\epsilon_0 y'}\hat{y} \quad , \quad \bar{B}' = 0$$

Now, we transform the Electric and magnetic field into the lab frame:

Griffiths eq (12.108)

$$E_x = E'_x = 0$$

$$E_y = \gamma(E'_y + vB'_z) = \gamma\frac{\lambda}{2\pi\epsilon_0 y}\hat{y}$$

$$E_z = \gamma(E'_z - vB'_y) = 0$$

$$B_x = B_x' = 0$$

$$B_y = \gamma(B_y' - v/c^2\, E_z') = 0$$

$$B_z = \gamma(B_z' + v/c^2\, E_y') = 0$$

Thus,

$$\boxed{F^{\mu\nu} = \begin{pmatrix} 0 & 0 & -\frac{\gamma\lambda}{2\pi\epsilon_0 c y} & 0 \\ 0 & 0 & -\frac{\gamma\beta\lambda}{2\pi\epsilon_0 c y} & 0 \\ \frac{\gamma\lambda}{2\pi\epsilon_0 c y} & \frac{\gamma\beta\lambda}{2\pi\epsilon_0 c y} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

#7.

$$\bar{F} = e\bar{E} \Rightarrow \frac{dp}{dt} = eE$$

$$\Rightarrow \frac{d}{dt} m\gamma\beta c = eE$$

$$\frac{d}{dt} m \frac{\beta c}{\sqrt{1-\beta^2}} = eE \qquad \beta \equiv \tanh\eta$$

$$\frac{d}{dt} mc \frac{\tanh\eta}{\sqrt{1-\tanh^2\eta}} = eE$$

$$\frac{d}{dt} mc \frac{\tanh\eta}{\operatorname{sech}\eta} = eE$$

$$\int d(mc\sinh\eta) = \int eE\, dt$$

$$mc\sinh\eta = eEt + C$$

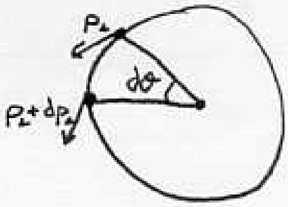
at $t=0$, $\beta=0$ thus:

$\tanh \eta = 0 \Rightarrow \sinh \eta = 0$ So, $C=0$

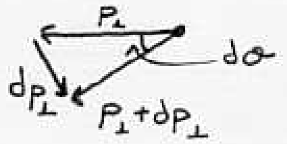
Now, $mc \sinh \eta = eEt$

Or, $$\boxed{\eta = \sinh^{-1}\left(\frac{eEt}{mc}\right)}$$

\# 8. (See Griffiths Example 12.11)



motion in the perpendicular plane $\perp$



• dp equals approximately the arc length: ① $dP_\perp \approx P_\perp d\theta$

Thus,

$$F_{cent} = \frac{dP_\perp}{dt} \overset{\text{using eq ①}}{=} P_\perp \frac{d\theta}{dt} = P_\perp \Omega = \gamma_\perp m_\perp \beta_\perp c \, \Omega$$

where, $$\boxed{\gamma_\perp \equiv \frac{1}{\sqrt{1-\beta_\perp^2}}}$$

• Note that we did not use mass $= m$ since the $\perp$ plane is not at rest with respect to the lab frame. When the particle is a rest in the $\perp$ plane then it has mass $= m_\perp$.

$$F_{cent.} = F_{Lorentz} \Rightarrow \gamma_\perp m_\perp \beta_\perp c \, \Omega = e v_\perp B$$

$$\Rightarrow \gamma_\perp m_\perp \beta_\perp c \, \Omega = e c \beta_\perp B$$

$$\boxed{\Omega = \frac{eB}{\gamma_\perp m_\perp}}$$

Now, in the lab frame:

$$p = (E, p_0, p_\perp \cos\theta, p_\perp \sin\theta)$$

We can express E and $p_\perp$ as follows:

$$E = \gamma mc = \gamma_\perp m_\perp c$$
$$p_\perp = \gamma m \beta_\perp c = \gamma_\perp m_\perp \beta_\perp c$$

Thus, $p \cdot p = E^2 - p_0^2 - p_\perp^2 = m^2c^2$

$$\gamma_\perp^2 m_\perp^2 c^2 - p_0^2 - \gamma_\perp^2 m_\perp^2 \beta_\perp^2 c^2 = m^2c^2$$

$$\gamma_\perp^2 m_\perp^2 c^2 (1 - \beta_\perp^2) = m^2c^2 + p_0^2$$

So, $\boxed{m_\perp = \sqrt{m^2 + p_0^2/c^2}}$

University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

Problem Set 8

1. Griffiths 11.3.

2. Griffiths 11.4. You need calculate only the **time average** Poynting vector, intensity, and total power radiated (this is much simpler than computing the full time-dependent expressions).

3. Griffiths 11.9.

4. Griffiths 11.13.

5. Griffiths 11.16. You need calculate only $dP/d\Omega$, not P (thereby avoiding a messy integral).

6. Griffiths 11.17 (a) and (b) only.

7. Griffiths 11.25.

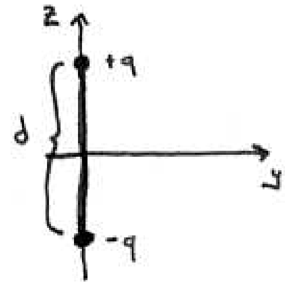
8. Griffiths 11.31.

Physics 110B
Homework #8

#1. (Griffiths 11.3)

$$\vec{I}(t) = \frac{dq}{dt}\hat{z} = -q_0\omega\sin\omega t\,\hat{z} \quad (\text{eq } 11.15)$$

$$p_0 \equiv q_0 d$$



Thus,

$$P = I^2 R = q_0^2\omega^2\sin^2(\omega t)\,R \quad (\text{eq } 11.15)$$

$$\Rightarrow \langle P\rangle_{\text{dissipated by wire}} = \tfrac{1}{2}q_0^2\omega^2 R$$

Equating this to the average power radiated eq (11.22):

$$\Rightarrow \langle P\rangle_{\text{radiated}} = \frac{\mu_0 p_0^2\omega^4}{12\pi c} = \frac{\mu_0 q_0^2 d^2\omega^4}{12\pi c} = \frac{1}{2}q_0^2\omega^2 R = \langle P\rangle_{\text{wire}}$$

$$\Rightarrow \boxed{R = \frac{\mu_0 d^2\omega^2}{6\pi c}} \qquad \text{since} \quad \omega = \frac{2\pi c}{\lambda}$$

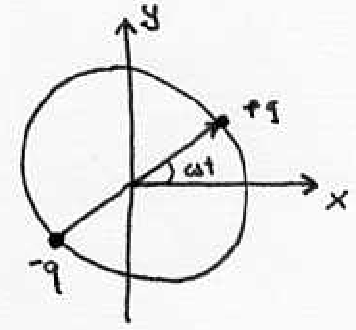
$$R = \frac{\mu_0 d^2}{6\pi c}\,\frac{4\pi^2 c^2}{\lambda^2} = \frac{2}{3}\pi\mu_0 c\left(\frac{d}{\lambda}\right)^2 = \frac{2}{3}\pi\sqrt{\frac{\mu_0}{\epsilon_0}}\left(\frac{d}{\lambda}\right)^2$$

$$\Rightarrow \boxed{R = 789.6\left(\tfrac{d}{\lambda}\right)^2\Omega}$$

For wires in an ordinary radio w/ $d = .05\,m$ and say $\lambda = 10^3\,m$:

$R = 789.6\left(\frac{.05}{10^3}\right)^2 = 2\times10^{-6}\,\Omega$ which is negligable compared to wire resistances.

#2. (Griffiths 11.4)



$$\bar{p} = p_0 (\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y})$$

For a single dipole on the z-axis we have the result given by (Eq. 11.18):

$$① \quad \bar{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos(\omega(t - r/c))\hat{\theta} \qquad (\text{Eq } 11.18)$$

Since, $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$ and, $\cos\theta = \frac{z}{r}$ in spherical coord.

thus, $\hat{\theta} = \left(\hat{z} - \frac{z}{r}\hat{r}\right)\left(-\frac{1}{\sin\theta}\right)$

Using this result in eq ① we get:

$$\bar{E} = \frac{\mu_0 p_0 \omega^2}{4\pi r}\cos(\omega(t - r/c))\left(\hat{z} - \frac{z}{r}\hat{r}\right)$$

Now, by superposition the rotating dipole will give the following Electric field:

$$\bar{E} = E_x + E_y = \frac{\mu_0 p_0 \omega^2}{4\pi r}\left(\cos(\omega(t - r/c))\left(\hat{x} - \frac{x}{r}\hat{r}\right) + \sin(\omega(t - r/c))\left(\hat{y} - \frac{y}{r}\hat{r}\right)\right)$$

From eq (11.19) we have for a single dipole in the z-direction a magnetic field:

$$\bar{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos(\omega(t - r/c))\hat{\varphi} \quad (\text{Eq } 11.19)$$

$$= \frac{1}{c}(\hat{r} \times \bar{E}) \quad \text{by looking at eq ①}$$

Thus,

$$\boxed{\begin{aligned}\bar{E}_{total} &= \frac{\mu_0 P_0 \omega^2}{4\pi r}\left(\cos(\omega(t-r/c))\left(\hat{x} - \frac{x}{r}\hat{r}\right) + \sin(\omega(t-r/c))\left(\hat{y} - \frac{y}{r}\hat{r}\right)\right) \\ \bar{B}_{total} &= \frac{1}{c}\left(\hat{r} \times \bar{E}_{total}\right)\end{aligned}}$$

Now, from the above results we can get the Poynting vector:

$$③ \quad \bar{S} = \frac{1}{\mu_0}\left(\bar{E} \times \bar{B}\right) = \frac{1}{\mu_0 c}\left(\bar{E} \times (\hat{r} \times \bar{E})\right) = \frac{1}{\mu_0 c}\left(E^2\hat{r} - (\bar{E}\cdot\hat{r})\bar{E}\right)$$

But,

$$\bar{E}\cdot\hat{r} = \frac{\mu_0 P_0 \omega^2}{4\pi r}\left(\cos(\omega t_r)\left(\hat{x}\cdot\hat{r} - \frac{x}{r}\right) + \sin(\omega t_r)\left(\hat{y}\cdot\hat{r} - \frac{y}{r}\right)\right)$$

$$\hat{x}\cdot\hat{r} = \left(\sin\theta\cos\varphi\,\hat{r} + \cos\theta\cos\varphi\,\hat{\theta} - \sin\varphi\,\hat{\varphi}\right)\cdot\hat{r} = \sin\theta\cos\varphi$$

$$\hat{x}\cdot\hat{r} = x/r$$

$$\hat{y}\cdot\hat{r} = \left(\sin\theta\sin\varphi\,\hat{r} + \cos\theta\sin\varphi\,\hat{\theta} + \cos\varphi\right)\cdot\hat{r} = \sin\theta\sin\varphi$$

$$\hat{y}\cdot\hat{r} = y/r$$

Thus,

$$\bar{E}\cdot\hat{r} = 0 \quad ④$$

Placing eq ④ in eq ③ and taking the average we have:

$$\boxed{\langle\bar{S}\rangle = \frac{\langle E^2\rangle}{\mu_0 c}\hat{r}}$$

Now, we need to find the Electric field squared:

$$\langle E^2\rangle = \left(\frac{\mu_0 P_0 \omega^2}{4\pi r}\right)^2\left\langle a^2\cos^2(\omega t_r) + b^2\sin^2(\omega t_r) + 2a\cdot b\,\sin(\omega t_r)\cdot\cos(\omega t_r)\right\rangle$$

$$\langle E^2 \rangle = \left(\frac{\mu_o p_o \omega^2}{4\pi r}\right)^2 \frac{1}{2}(a^2+b^2) \quad (5)$$

$\bar{a} \equiv \hat{x} - (x/r)\hat{r}$; $\bar{b} \equiv \hat{y} - (y/r)\hat{r}$, thus: $\hat{r}\cdot\hat{x} = x/r$

$$a^2 = (\hat{x} - (\tfrac{x}{r})\hat{r})\cdot(\hat{x} - (\tfrac{x}{r})\hat{r}) = 1 + \frac{x^2}{r^2} - 2\frac{x^2}{r^2} = 1 - \frac{x^2}{r^2}$$

$$b^2 = 1 - y^2/r^2$$

Placing the above results into eq (5):

$$\langle E^2 \rangle = \left(\frac{\mu_o p_o \omega^2}{4\pi r}\right)^2 \frac{1}{2}\left(2 - \frac{x^2+y^2}{r^2}\right)$$

$$= \left(\frac{\mu_o p_o \omega^2}{4\pi r}\right)^2 \frac{1}{2}\left(2 - \frac{r^2\sin^2\theta\cos^2\varphi + r^2\sin^2\theta\sin^2\varphi}{r^2}\right)$$

$$= \left(\frac{\mu_o p_o \omega^2}{4\pi r}\right)^2 \left(1 - \frac{\sin^2\theta}{2}\right)$$

Thus,

$$\boxed{\langle \bar{S} \rangle = \frac{1}{\mu_o c}\left(\frac{\mu_o p_o \omega^2}{4\pi r}\right)^2\left(1 - \frac{\sin^2\theta}{2}\right)\hat{r}}$$

$$\boxed{\text{Intensity} = |\langle \bar{S} \rangle| = \frac{1}{\mu_o c}\left(\frac{\mu_o p_o \omega^2}{4\pi r}\right)^2\left(1 - \frac{\sin^2\theta}{2}\right)}$$

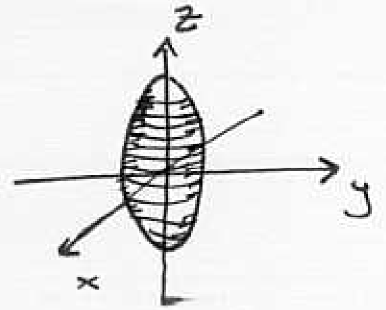
Power equals,

$$P = \int \langle \bar{S} \rangle \cdot d\bar{a} = \frac{\mu_o}{c}\left(\frac{p_o \omega^2}{4\pi}\right)^2 \int \frac{1}{r^2}\left(1 - \frac{\sin^2\theta}{2}\right) r^2 \sin\theta \, d\theta \, d\varphi$$

$$= \frac{\mu_o p_o^2 \omega^4}{16\pi^2 c} 2\pi\left(\int_0^\pi \sin\theta \, d\theta - \frac{1}{2}\int_0^\pi \sin^3\theta \, d\theta\right) = \frac{\mu_o p_o^2 \omega^4}{16\pi^2 c} 2\pi\left(2 - \frac{1}{2}\frac{4}{3}\right)$$

Intensity profile

$I \sim 1 - \frac{1}{2}\sin^2\theta$



$$\boxed{P_{radiated} = \frac{\mu_0 p_0^2 \omega^4}{6\pi c}}$$

For a single dipole the total power radiated equals

$P_{single\ dipole} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$ (eq 11.22), thus the power radiated for the superposition of two dipoles will be the two individual dipole powers plus any power from the cross term.

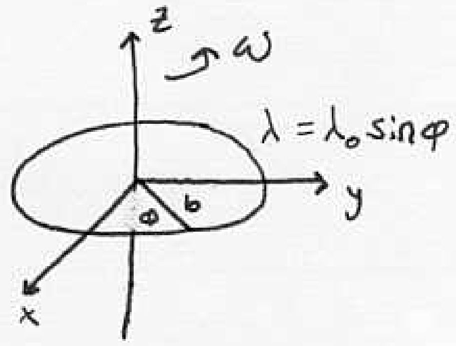
$$P_{total} = P_{x\text{-dipole}} + P_{y\text{-dipole}} + P_{cross\ term}$$

or $\bar{S}_{total} = \frac{1}{\mu_0}(\bar{E}_T \times \bar{B}_T) = \frac{1}{\mu_0}\left((E_x + E_y)\times(B_x + B_y)\right)$

$$= \frac{1}{\mu_0}\Big(E_x \times B_x + E_y \times B_y + \underbrace{\cancelto{0}{E_x \times B_y + E_y \times B_x}}_{\text{cross terms}}\Big)$$

- In this particular case the components fields are 90° out of phase, so the cross terms go to zero, and the total power equals the sum of the two dipole powers.

#3. (Griffiths 11.9)



At time $t=0$ the dipole moment of the ring is

$$\bar{p}_0 = \int \lambda r\, dl = \int \lambda_0 \sin\varphi\, \vec{r}\, b\, d\varphi$$

$$= \lambda_0 \int b\sin\varphi\,(b\sin\varphi\,\hat{y} + b\cos\varphi\,\hat{x})\,d\varphi$$

$$= b^2\lambda_0 \int_0^{2\pi} \sin^2\varphi\, d\varphi\,\hat{y} + b^2\lambda_0 \int_0^{2\pi} \sin\varphi\cos\varphi\, d\varphi\,\hat{x}$$

$$\bar{p}_0 = \pi b^2 \lambda_0 \hat{y}$$

As it rotates counterclockwise.

$$\bar{p}(t) = p_0 (\cos\omega t\, \hat{y} - \sin\omega t\, \hat{x})$$

So,

$$\ddot{p} = -\omega^2 \bar{p}$$

Therefore, using (Eq. 11.60):

$$P = \frac{\mu_0 \ddot{p}^2}{6\pi c} = \frac{\mu_0}{6\pi c} \omega^4 p_0^2 = \frac{\mu_0}{6\pi c} \omega^4 \pi^2 b^4 \lambda_0^2$$

$$\boxed{P = \frac{\pi \mu_0 \omega^4 b^4 \lambda_0^2}{6c}}$$

#4. (Griffiths 11.13)

(a) Using the Larmor formula,

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \quad (\text{eq } 11.70)$$

The time it takes to come to rest is $t = v_0/a$, so energy radiated equals,

$$U_{rad} = Pt = \frac{\mu_0 q^2 a^2}{6\pi c} \frac{v_0}{a} = \frac{\mu_0 q^2 a v_0}{6\pi c}$$

The initial kinetic energy was,

$$U_{kin_0} = \tfrac{1}{2} M v_0^2 \Rightarrow \text{Thus, the fraction radiated} = \frac{U_{rad}}{U_{kin_0}} = \boxed{\frac{\mu_0 q^2 a}{3\pi M v_0 c}}$$

(b) $V_0 = 10^5$ m/s $\quad d = 30$ Å

From basic kinematics,

$$V_0^2 = 2ad \implies a = V_0^2/2d$$

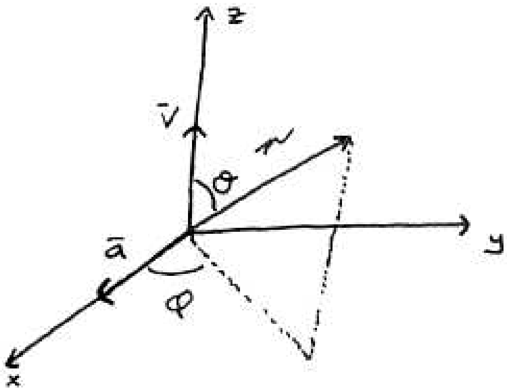
Thus,

$$f = \frac{\mu_0 q^2}{3\pi M V_0 c}\,\frac{V_0^2}{2d} = \frac{\mu_0 q^2 V_0}{6\pi M c d} = \frac{(4\pi\times10^{-7})(1.6\times10^{-19})^2(10^5)}{6\pi(9.11\times10^{-31})(3\times10^{8})(3\times10^{-9})}$$

$$\boxed{f = 2\times10^{-10}}$$

So, the radiative losses in an ordinary wire are negligible.

#5. (Griffiths 11.16)



$\bar{v} = v\hat{z}$, $\quad \bar{a} = a\hat{x}$

$\hat{\mathscr{r}} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$

The Lienard formula is:

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0}\,\frac{(\hat{\mathscr{r}}\times(\bar{u}\times\bar{a}))^2}{(\hat{\mathscr{r}}\cdot\bar{u})^5} \quad \text{(Eq 11.72)} \qquad \beta \equiv \frac{v}{c}$$

$$\bar{u} = c\hat{\mathscr{r}} - \bar{v} = c\hat{\mathscr{r}} - v\hat{z}$$

$$\Rightarrow \hat{\mathscr{r}}\cdot\bar{u} = c - v(\hat{\mathscr{r}}\cdot\hat{z}) = c - v\cos\theta = c\left(1 - \tfrac{v}{c}\cos\theta\right) = c\left(1 - \beta\cos\theta\right)$$

$$\bar{a}\cdot\bar{u} = ac(\hat{x}\cdot\hat{\mathscr{r}}) - av\cancelto{0}{(\hat{x}\cdot\hat{z})} = ac\sin\theta\cos\varphi$$

$$u^2 = \bar{u}\cdot\bar{u} = c^2 - 2cv(\hat{\mathscr{r}}\cdot\hat{z}) + v^2 = c^2 + v^2 - 2cv\cos\theta$$

Now, using the above results we can reduce the Lienard formula:

$$\hat{r}\times(\bar{u}\times\bar{a}) = (\hat{r}\cdot\bar{a})\,\bar{u} - (\hat{r}\cdot\bar{u})\,\bar{a}$$

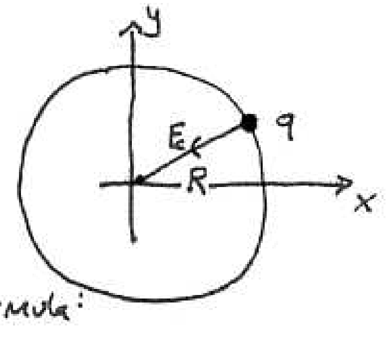
$$\left(\hat{r}\times(\bar{u}\times\bar{a})\right)^2 = (\hat{r}\cdot\bar{a})^2u^2 - 2(\bar{u}\cdot\bar{a})(\hat{r}\cdot\bar{a})(\hat{r}\cdot\bar{u}) + (\hat{r}\cdot u)^2a^2$$

$$= (c^2+v^2-2cv\cos\theta)(a\sin\theta\cos\varphi)^2 - 2(ac\sin\theta\cos\varphi)(a\sin\theta\cos\varphi)c(1-\beta\cos\theta)$$
$$+a^2c^2(1-\beta\cos\theta)^2$$
$$= a^2\left(c^2(1-\beta\cos\theta)^2 + (\sin^2\theta\cos^2\varphi)(c^2+v^2-2cv\cos\theta-2c^2+2cv\cos\theta)\right)$$
$$= a^2c^2\left((1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\varphi\right)$$

Thus, placing the above results into the Lienard formula:

$$\frac{dP}{d\Omega} = \frac{q^2a^2c^2}{16\pi^2\epsilon_0}\frac{\left((1-\beta\cos\theta)^2-(1-\beta^2)\sin^2\theta\cos^2\varphi\right)}{c^5\,(1-\beta\cos\theta)^5}$$

$$= \boxed{\frac{\mu_0q^2a^2}{16\pi^2c}\frac{\left((1-\beta\cos\theta)^2-(1-\beta^2)\sin^2\theta\cos^2\varphi\right)}{(1-\beta\cos\theta)^5}}$$



#6. (Griffiths 11.17)

(a) The radiation will apply a reaction force given by the Abraham-Lorentz formula:

$$\bar{F}_{rad} = \frac{\mu_0q^2}{6\pi c}\dot{\bar{a}} \quad \text{(eq 11.80)}$$

Thus, to counter act this force we need to apply a force,

$$\bar{F}_e = -\frac{\mu_0q^2}{6\pi c}\dot{\bar{a}}$$

$$\vec{a}_{cent} = -\omega^2 \vec{r}$$

$$\Rightarrow \dot{\vec{a}} = -\omega^2 \vec{v}$$

So,

$$\boxed{\vec{F}_e = \frac{\mu_0 q^2}{6\pi c} \omega^2 \vec{v}}$$

The extra power delivered is,

$$P_e = \vec{F}_e \cdot \vec{v} = \boxed{\frac{\mu_0 q^2}{6\pi c} \omega^2 v^2 = P_e}$$

The power radiated from the Larmor formula is,

$$\text{(Eq. 11.70)} \quad P_{rad} = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{\mu_0 q^2}{6\pi c} \omega^4 r^2 = \frac{\mu_0 q^2}{6\pi c} \omega^2 v^2$$

Thus, we see that the two expressions are in fact equal $P_e = P_{rad}$

(b) For simple harmonic motion,

$$\vec{r}(t) = A \cos \omega t \, \hat{z}$$

$$\vec{v}(t) = \dot{\vec{r}} = -A\omega \sin \omega t \, \hat{z}$$

$$\vec{a}(t) = \dot{\vec{v}} = -A\omega^2 \cos \omega t \, \hat{z} = -\omega^2 \vec{r}$$

Thus,

$$\boxed{\vec{F}_e = \frac{\mu_0 q^2}{6\pi c} \omega^2 \vec{v}, \quad P_e = \frac{\mu_0 q^2}{6\pi c} \omega^2 v^2}$$

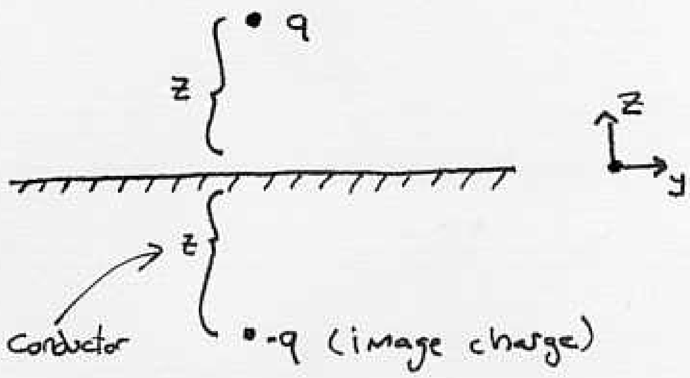
However, this time $a^2 = \omega^4 r^2 = \omega^4 A^2 \cos^2 \omega t$

Whereas $\omega^2 v^2 = \omega^4 A^2 \sin^2\omega t$

Thus, $$\boxed{P_{rad} = \frac{\mu_0 q^2}{6\pi c}\omega^4 A^2 \cos^2\omega t \neq P_e = \frac{\mu_0 q^2}{6\pi c}\omega^4 A^2 \sin^2\omega t}$$

The power you deliver is not equal to the power radiated. However, since the time averages of $\sin^2\omega t$ and $\cos^2\omega t$ are 1/2 over a full cycle the energy radiated is the same as the energy input.

#7. (Griffiths 11.25)



$$\bar{p}(t) = 2q\bar{z}(t)$$

$$\ddot{p} = 2q\ddot{z}$$

The charge above the conductor is attracted to the image charge through a Coulombic force,

$$F_{col} = M\ddot{z} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2z)^2}$$

$$\Rightarrow \ddot{z} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4Mz^2} = -\frac{\mu_0 c^2 q^2}{16\pi M z^2}$$

$$\Rightarrow \ddot{p} = -\frac{\mu_0 c^2 q^3}{8\pi M z^2}$$

Using (Eq 11.60), the power radiated is

$$\text{(Eq 11.60)} \quad P_{rad} = \frac{\mu_0 \ddot{p}^2}{6\pi c} = \frac{\mu_0}{6\pi c}\left(-\frac{\mu_0 c^2 q^3}{8\pi M z^2}\right)^2$$

$$= \frac{\mu_0^3 c^3 q^6}{6(4\pi)^3 M^2 z^4} = \boxed{\left(\frac{\mu_0 c q^2}{4\pi}\right)^3 \frac{1}{6M^2 z^4} = P_{rad}}$$

#8. (Griffiths 11.31)

(a) $$W_{\text{hyperbolic trajectory}} = \sqrt{b^2 + c^2t^2} \quad (\text{Eq. } 10.45)$$

$$P_{rad} = \frac{\mu_0 q^2 a^2 \gamma^6}{6\pi c} \quad (\text{Eq. } 11.75)$$

So, $$v = \dot{w} = \frac{c^2 t}{\sqrt{b^2 + c^2t^2}}$$

$$a = \dot{v} = \frac{c^2}{\sqrt{b^2+c^2t^2}} - \frac{c^4t^2}{(b^2+c^2t^2)^{3/2}} = \frac{c^2}{(b^2+c^2t^2)^{3/2}}(b^2+c^2t^2-c^2t^2)$$

$$= \frac{b^2c^2}{(b^2+c^2t^2)^{3/2}}$$

Now, $$\gamma^2 = \frac{1}{1 - v^2/c^2} = \frac{1}{1 - c^2t^2/(b^2+c^2t^2)} = \frac{b^2+c^2t^2}{b^2+c^2t^2-c^2t^2}$$

$$= 1/b^2 \, (b^2 + c^2t^2)$$

So $$P_{rad} = \frac{\mu_0 q^2}{6\pi c} \frac{b^4c^4}{(b^2+c^2t^2)^3} \frac{(b^2+c^2t^2)^3}{b^6} = \boxed{\frac{q^2 c}{6\pi\epsilon_0 b^2} = P_{rad}}$$

$\boxed{\text{Yes, a particle in hyperbolic motion radiates}}$

(b) $$F_{rad} = \frac{\mu_0 q^2 \gamma^4}{6\pi c}\left(\dot{a} + \frac{3\gamma^2 a^2 v}{c^2}\right)$$ (problem 11.30)

$$\dot{a} = -\frac{3}{2}\,\frac{b^2c^2(2c^2t)}{(b^2+c^2t^2)^{5/2}} = -\frac{3b^2c^4t}{(b^2+c^2t^2)^{5/2}}$$

Thus, $$F_{rad} = \frac{\mu_0 q^2\gamma^4}{6\pi c}\left(-\frac{3b^2c^4t}{(b^2+c^2t^2)^{5/2}} + \frac{3}{c^2}\,\frac{(b^2+c^2t^2)}{b^2}\,\frac{b^4c^4}{(b^2+c^2t^2)^3}\,\frac{c^2t}{\sqrt{b^2+c^2t^2}}\right)$$

$\Rightarrow$ $\boxed{F_{rad} = 0}$ $\boxed{\text{No, the radiation reaction is zero}}$

University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

Problem Set 9

1. A general Jones vector describing a fully coherent electromagnetic wave with a nonzero x polarized component can be written as

$$\frac{1}{\sqrt{1+b^2}}\begin{pmatrix} 1 \\ be^{i\delta} \end{pmatrix} ,$$

where b is real.
(**a**). Show that this represents elliptically polarized light in which the major axis of the ellipse makes an angle

$$\tfrac{1}{2}\arctan\left(\frac{2b\cos\delta}{1-b^2}\right)$$

with the x axis.
(**b**). How can you tell whether the light is right-hand or left-hand elliptically polarized?
(**c**). Show that elliptically polarized light can be written as a sum of linearly and circularly polarized light. What is the relationship between the major axis of the ellipse and the axis along which its linearly polarized component is polarized?

2. Pedrotti×2 14-5. (In their notation, x is horizontal and y is vertical. You can easily do the problem without their hint, which seems not to be of much help.)

3. Pedrotti×2 14-12. ("OA" signifies "slow axis", and "TA" signifies "transmission axis". They have in mind the reflection that occurs when, after passing through the isolator, the light enters a material with real refractive index $n > 1$.)

4. Pedrotti×2 14-22.

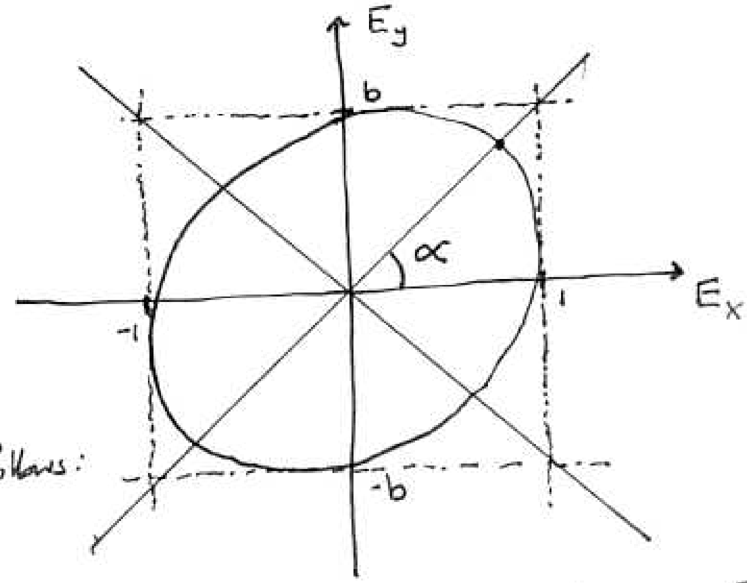
5. Calculate the interference pattern that would be obtained if four identical slits instead of two were used in Young's experiment. (Assume equal spacing of the slits). Make a rough plot.

6. Pedrotti×2 12-11.

7. Pedrotti×2 12-13.

8. Pedrotti×2 12-14.

Physics 110B
Homework # 9



#1. (a) The jones vector for the ellipitically polarized light is given by:

$$\frac{1}{\sqrt{1+b^2}}\begin{pmatrix} 1 \\ be^{i\delta} \end{pmatrix}$$

The components can be written as follows:

On the major axis : $\tan\alpha = \dfrac{E_y}{E_x}$

$$E_x = \hat{x}\exp(i(\omega t - kx))$$

$$E_y = b\hat{y}\exp(i(\omega t - kx + \delta))$$

Or, looking at the real part of the above equations:

$$E_x = \cos(\omega t - kx) \quad , \quad E_y = b\cos(\omega t - kx + \delta)$$

$$\Rightarrow E_y = b\cos(\omega t - kx)\cos\delta - b\sin(\omega t - kx)\sin\delta$$

$$= bE_x\cos\delta - b\sin(\omega t - kx)\sin\delta$$

$$= bE_x\cos\delta - b\left(1 - E_x^2\right)^{1/2}\sin\delta$$

$$\Rightarrow \boxed{\frac{E_y^2}{b^2} + E_x^2 - \frac{2E_yE_x}{b}\cos\delta = \sin^2\delta}$$

Pedrotti (Eq. 14-12)

The above equation describes an ellipse as given in the above figure.

The ellipse in the figure is contained in a rectangle with sides 2 and 2b. The position of the major axis of the ellipse, at an angle α to the x-axis, is found as follows:

- The amplitude $E_x^2 + E_y^2$ is max. on the major axis. Therefore at this point

$$\Delta(\underbrace{E_x^2 + E_y^2}_{\equiv f}) = \frac{\partial f}{\partial E_y} \partial E_y + \frac{\partial f}{\partial E_x} \partial E_x = 0$$

$$① \Rightarrow E_y \, \partial E_y + E_x \, \partial E_x = 0$$

Now, similarly we maximize the equation for the ellipse above

$$\Delta(\underbrace{E_y^2/b^2 + E_x^2 - \frac{2E_y E_x}{b} \cos\delta}_{\equiv g}) = \frac{\partial g}{\partial E_x} \partial E_x + \frac{\partial g}{\partial E_y} \partial E_y = 0$$

$$② \Rightarrow \left(E_x - \frac{\cos\delta}{b} E_y\right) \partial E_x + \left(\frac{E_y}{b^2} - \frac{\cos\delta}{b} E_x\right) \partial E_y = 0$$

Combining eq① and eq② we have using the fact that $\tan\alpha = \dfrac{E_y}{E_x}$

$$1 - \frac{\cos\delta}{b} \tan\alpha = \frac{1}{b^2} - \frac{\cos\delta}{b} \cot\alpha$$

Since $\tan\alpha - \cot\alpha = \cot 2\alpha$, we find:

$$\boxed{\tan 2\alpha = \frac{2b\cos\delta}{1 - b^2}}$$

Pedrotti
(Eq 14-10)

(b) Using the results from Table 14-1 (p. 288) we see the sign of the imaginary term determines the handedness the light:

$$\frac{1}{\sqrt{1+b^2}}\begin{pmatrix}1\\ be^{i\delta}\end{pmatrix} = \frac{1}{\sqrt{1+b^2}}\begin{pmatrix}1\\ b\cos\delta + i\,b\sin\delta\end{pmatrix}$$

If	$b\sin\delta > 0$	then it is left-handed
If	$b\sin\delta < 0$	then the light is right-handed

(c) We break the Jones vector into linear and elliptical terms:

$$\frac{1}{\sqrt{1+b^2}}\begin{pmatrix}1\\ be^{i\delta}\end{pmatrix} = \frac{1}{\sqrt{1+b^2}}\left(\begin{pmatrix}1-b\sin\delta\\ b\cos\delta\end{pmatrix} + \begin{pmatrix}b\sin\delta\\ i\,b\sin\delta\end{pmatrix}\right)$$

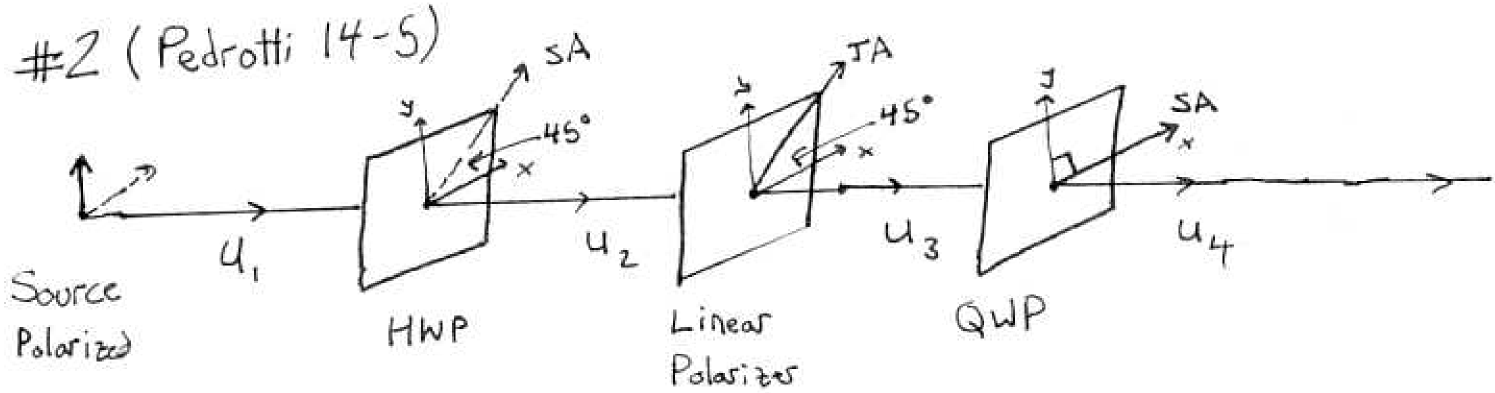
$$\boxed{= \underbrace{\frac{1}{\sqrt{1+b^2}}\begin{pmatrix}1-b\sin\delta\\ b\cos\delta\end{pmatrix}}_{\text{linearly polarized}} + \underbrace{\frac{b\sin\delta}{\sqrt{1+b^2}}\begin{pmatrix}1\\ i\end{pmatrix}}_{\text{circularly polarized}}}$$

Linearly polarized component is polarized in the direction:

$$\tan\beta = \frac{b\cos\delta}{1-b\sin\delta} \quad \Leftarrow \text{angle of polarization}$$

Hence,
$$\boxed{\frac{\tan 2\alpha}{\tan\beta} = \frac{2-2b\sin\delta}{1-b^2}}$$

#2 (Pedrotti 14-5)



(a) The original light is represented by the following Jones vector:

$$u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Before going through the HWP we rotate the light $-\pi/4$ and after coming out again we rotatate the light $+\pi/4$:

$$u_2 = M_{\text{rotate} +\pi/4}\, M_{HWP}\, M_{\text{rotate} -\pi/4}\, u_1$$

$$= \begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix} \begin{pmatrix} e^{-i\pi/2} & 0 \\ 0 & -e^{-i\pi/2} \end{pmatrix} \begin{pmatrix} \cos(-\pi/4) & -\sin(-\pi/4) \\ \sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix} u_1$$

$$= \frac{e^{-i\pi/2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} u_1$$

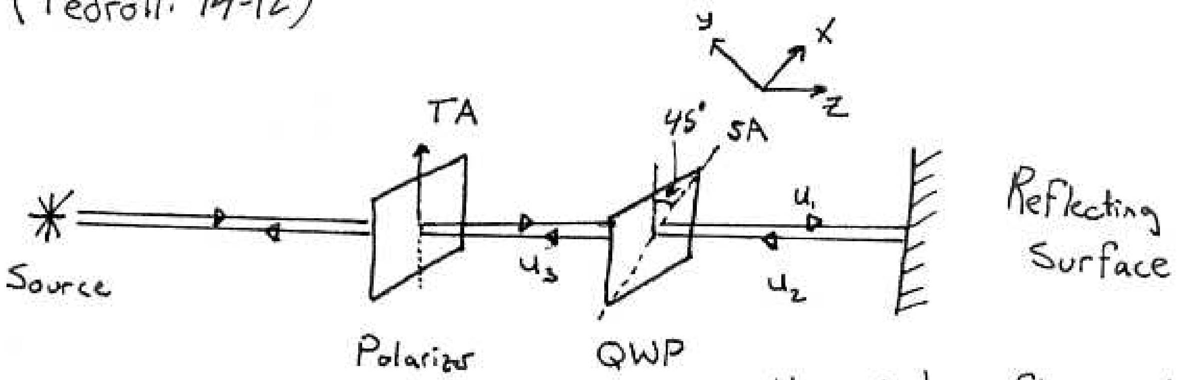
$$= \frac{e^{-i\pi/2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} u_1 = \frac{e^{-i\pi/2}}{2} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} u_1 = \boxed{e^{-i\pi/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = u_2}$$

(b) $$u_3 = M_{\substack{\text{linear} \\ \text{polarizer}}}\, u_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} e^{-i\pi/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\frac{e^{-i\pi/2}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = u_3}$$

(c) $$u_4 = M_{QWP}\, u_3 = e^{i\pi/4} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \frac{e^{-i\pi/2}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boxed{\frac{e^{-i\pi/4}}{2} \begin{pmatrix} 1 \\ -i \end{pmatrix} = u_4}$$

The final product is right circularly polarized light.

#3. (Pedrotti: 14-12)



u_1 = vector after; polarizer, QWP
u_2 = vector after; reflection
u_3 = vector after; reflection, QWP

We make the slow axis of the QWP parallel to the x-axis.
This means the polarized light coming from the polarizer can be represented by the vector $E_{pol} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

The QWP can be represented by the Jones matrix:

$$M_{QWP} = e^{i\pi/4}\begin{pmatrix}1 & 0\\0 & -i\end{pmatrix}$$

Thus, the light coming out of the QWP can be expressed as,

$$u_1 = M_{QWP}\,E_{pol} = \begin{pmatrix}1 & 0\\0 & -i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

So, we get right circularly polarized light

Upon reflection, the wave undergoes a 180° phase shift. However, both components of $\bar{u}_1$ undergo the same shift so it is of the consequence.

$$u_2 = u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$ upon reflection still circularly polarized

Now, the wave goes again through the QWP and we get,

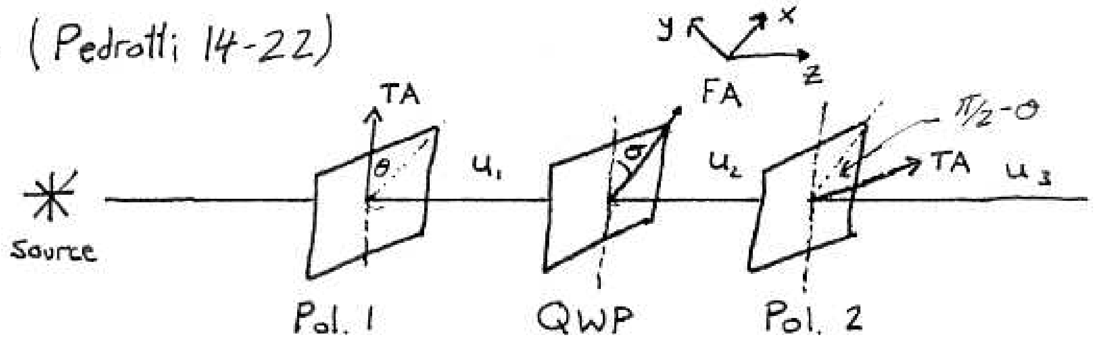
$$u_3 = M_{QWP}\,u_2 = \begin{pmatrix}1 & 0\\0 & -i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

Lastly, the light goes back through the linear polarizer.

$$u_{\text{light out pol}} = M_{\text{pol.}}\, u_3 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \boxed{0 = u_{\text{light out pol.}}}$$

Thus, we see that $\boxed{\text{no light is reflected back to the source.}}$

#4 (Pedrotti 14-22)



The x-axis has been chosen to be parallel to the fast axis of the QWP. Thus,

$$u_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \implies u_2 = M_{QWP}\, u_1 = e^{i\pi/4}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$u_2 = e^{i\pi/4}\begin{pmatrix} \cos\theta \\ i\sin\theta \end{pmatrix}$$

Now going through the last polarizer using (Eq. 14-5) for $M_{\text{pol 2}}$,

$$u_3 = M_{\text{pol 2}}\, u_2 = e^{i\pi/4}\begin{pmatrix} \cos^2(\pi/2-\theta) & \sin(\pi/2-\theta)\cos(\pi/2-\theta) \\ \sin(\pi/2-\theta)\cos(\pi/2-\theta) & \sin^2(\pi/2-\theta) \end{pmatrix} u_2$$

$$= e^{i\pi/4}\begin{pmatrix} \sin^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \cos^2\theta \end{pmatrix}\begin{pmatrix} \cos\theta \\ i\sin\theta \end{pmatrix} = e^{i\pi/4}\begin{pmatrix} \cos\theta\sin^2\theta + i\cos\theta\sin^2\theta \\ \cos^2\theta\sin\theta + i\cos^2\theta\sin\theta \end{pmatrix}$$

$$\Rightarrow \boxed{u_{\text{emerging light}} = e^{i\pi/4}(1+i)\begin{pmatrix} \cos\theta\sin^2\theta \\ \cos^2\theta\sin\theta \end{pmatrix}}$$

The intensity of the emerging light equals,

$$I = u_3^T u_3 = e^{-i\pi/4} e^{i\pi/4} (1-i)(1+i) (\cos\theta \sin^2\theta, \cos^2\theta \sin\theta) \begin{pmatrix} \cos\theta \sin^2\theta \\ \cos^2\theta \sin\theta \end{pmatrix}$$

$$= 2 (\cos^2\theta \sin^4\theta + \cos^4\theta \sin^2\theta) = 2 \cos^2\theta \sin^2\theta$$

Thus, for initial intensity I_o thru the above arrangement,

$$\boxed{I_{\text{emerging light}} = I_o \, 2 \cos^2\theta \sin^2\theta}$$

#5 The problem of N slits is done in Pedrotti see pages 341-345

From pedrotti we have the result:

$$I = I_0 \left(\frac{\sin\beta}{\beta} \right)^2 \left(\frac{\sin N\alpha}{\sin\alpha} \right)^2 \qquad (\text{Eq } 16 \cdot 32)$$

for very narrow slits $\beta \to 0$ thus $\frac{\sin\beta}{\beta} \Rightarrow \frac{\cos\beta}{\beta} = 1$

① $$I = I_0 \left(\frac{\sin N\alpha}{\sin\alpha} \right)^2 = I_0 \left(\frac{\sin 4\alpha}{\sin\alpha} \right)^2$$ For $N=4$ slits

$\alpha = \frac{1}{2} ka \sin\theta \approx \frac{1}{2} ka\theta$ $\quad$ θ ~ being small

a ~ the slit separation

The maxima of the above equation are given by:

$\alpha = \frac{p\pi}{N}$ or, $a\sin\theta \frac{\pi}{\lambda} = \frac{p\pi}{N} \Rightarrow a\sin\theta = \frac{p\lambda}{N}$ $\quad p = 0, \pm 1, \pm 2 \ldots$

principal Maxima occur for $p = 0, \pm N, \pm 2N, \ldots$

Minima occur for = all other values

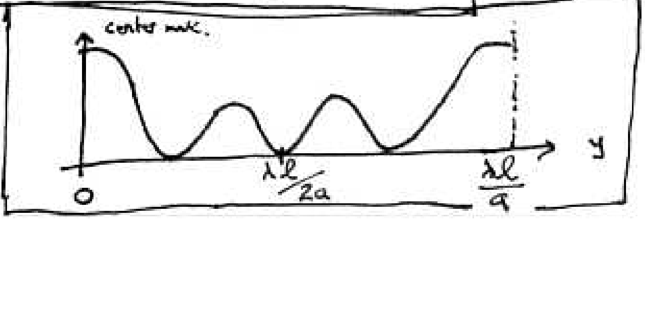
$\Rightarrow \frac{p\lambda}{Na} = \frac{y}{\ell} \Rightarrow y = \frac{p\lambda\ell}{Na}$ $\quad$ ℓ ~ distance from slit to screen

y ~ height of max. and min.

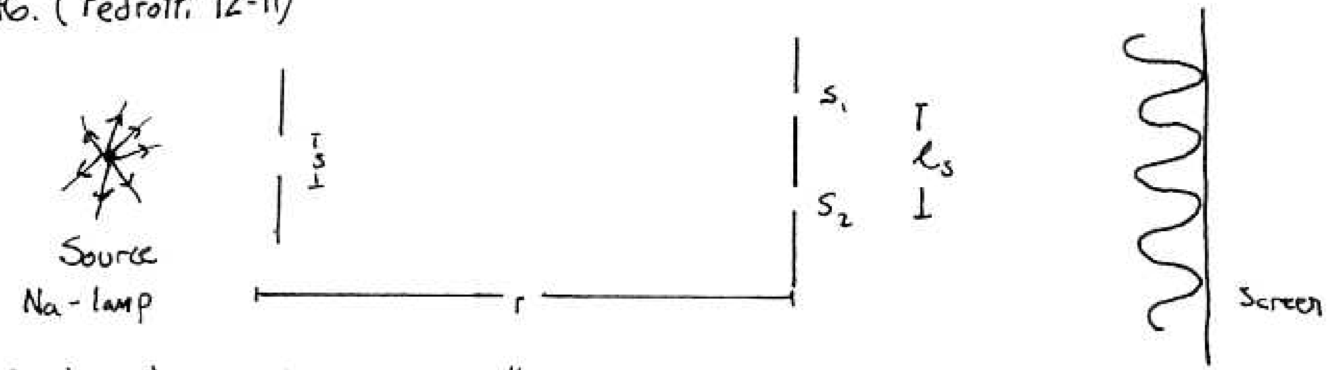
Thus, for the four slit Young's experiment we have:

$y = \frac{p\lambda\ell}{4a}$	max occur for $p = 0, \pm 4, \pm 8 \ldots$ min occur for = all other integer values

Thus, we will get the following intensity pattern



#6. (Pedrotti 12-11)



s = diameter = .5 mm = 5×10^{-4} m
$\lambda = 5890$ Å = 5.89×10^{-7} m

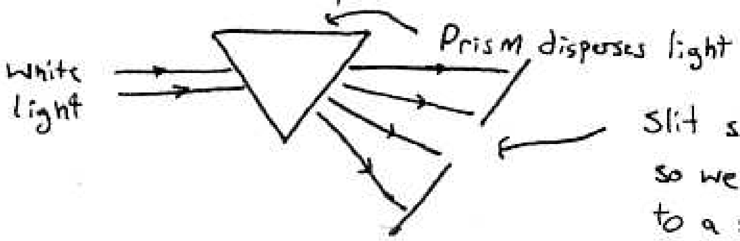
$r = 1$ m
$l_s^{max} = ?$

$$\Rightarrow \quad l_s^{max} = \frac{r\lambda}{s} = 1.178 \times 10^{-3}\ \text{m} \quad \text{(Eq. 12-35)}$$

$$\boxed{l_s^{Max} = .118\ \text{cm}}$$

#7. (Pedrotti 12-13)

(a) A simple example of a monochromator is a prism and slit.



Slit selects out a portion of the spectrum, so we go from a large range of wavelength (white light) to a smaller range.

The linear dispersion is a measure of how well the monochromator disperses the input light. It tells one, for a given slit width, the range of wavelengths one gets out:

Thus, $\lambda_o = 500$ nm
$w = 200\,\mu m$

linear dispersion = 2 nm/mm

$$\Longrightarrow \quad \delta\lambda = \underset{\text{dispers.}}{\text{linear}} \cdot w = .4\ \text{mm}$$

(eq 12-17) $\boxed{l_t = \frac{\lambda_o^2}{\delta\lambda} = 6.25 \times 10^{-4}\ \text{M}}$

$$\Rightarrow \boxed{\tau_o = \frac{\ell_T}{c} = 2.08\times10^{-12}\,s}$$

(b) OPD = .4 mm

$$\Rightarrow \tau = \frac{OPD}{c} = 1.33\times10^{-12}\,s$$

$$\Rightarrow |\gamma_{12}(\tau)| = 1 - \frac{\tau}{\tau_o} = \boxed{.361 = V} \quad (\text{Eq } 12\text{-}29)$$

(c) $I_{max} = 100$

I_{min} = Background Irradiance

$$V = .361 = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \Rightarrow I_{min} = 47 \quad (\text{Eq } 12\text{-}32)$$

$$\Rightarrow \boxed{\Delta I = 53}$$

#8. (Pedrotti 12-14)

The height of the cylinder is determined by the temporal coherence. The book gives the coherence length of white light (pg. 254):

$$h = \ell_T \cong 1000\,nm = 1.0\times10^{-6}\,m = 1.8\,\lambda_o \quad (\lambda_o = 550\,nm)$$

The spatial coherence can be found as in the last example:

$$(\text{Eq } 12\text{-}36) \quad \ell_s = \frac{1.22\,\lambda_o}{\theta} = \frac{1.22\cdot(5.5\times10^{-7})\,m}{8.727\times10^{-3}\,rad} = 7.69\times10^{-5}\,m$$

diameter of base $= (.25)\,\ell_s = 1.92\times10^{-5}\,m \;\rightarrow\; r = 9.61\times10^{-6}\,m$
$= 17.48\lambda_o$

$$\Rightarrow \boxed{\begin{array}{lll} h = 1.01\times10^{-6}\,m & & h = 1.83\lambda_o \\ A = \pi r^2 = 2.9\times10^{-10}\,m^2 & \text{or} & d = 34.96\lambda_o \end{array}}$$

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Physics 110B Spring 2001 Section 1 (*Strovink*)

Problem Set 10

1. Pedrotti×2 11-12.

2. Pedrotti×2 19-9.

3. Pedrotti×2 19-11.

4. Pedrotti×2 16-7.

5. Pedrotti×2 16-13.

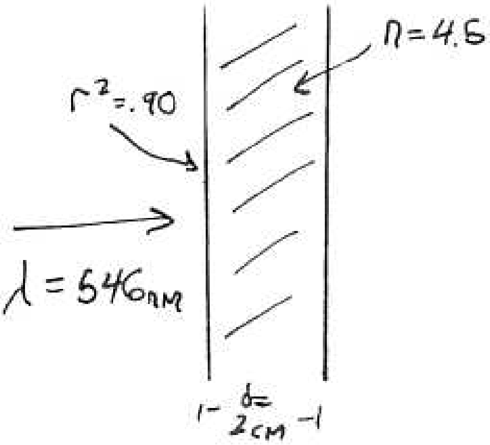
6. Pedrotti×2 16-22.

7. Pedrotti×2 16-25.

8. Pedrotti×2 17-8.

Physics 110B
Homework #10

#1 (Pedrotti 11-12)



(a) eq.(11-40)

$$m_{max} = \frac{2t}{\lambda/n} = \frac{2nd}{\lambda} = \frac{2\cdot 4.5\cdot 2cm}{5.46\times10^{-7}cm}$$

$$\boxed{m_{Max} = 329670}$$

(b) eq.(11-30)

$$F \equiv \frac{4r^2}{(1-r^2)^2} = 360$$

eq(11-32)

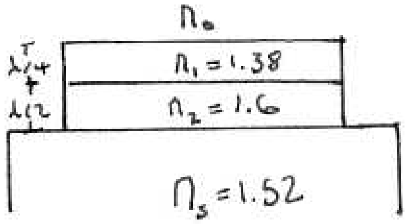
$$F = \frac{T_{max} - T_{min}}{T_{min}} = \frac{T_{Max}}{T_{Min}} - 1$$

$$\Rightarrow \boxed{\frac{T_{Max}}{T_{Min}} = 361}$$

(c)

$$\mathcal{R} = \frac{\pi}{2} m\sqrt{F} = \frac{\pi}{2}(329670)\sqrt{361} = \boxed{9.83\times10^{6} = \mathcal{R}}$$

#2 (Pedrotti 19-9)



eq.(19-24)

$$\mathcal{M} = \begin{bmatrix} \cos\delta & i\sin\delta/\gamma_1 \\ i\gamma_1\sin\delta & \cos\delta \end{bmatrix}$$

$$\lambda/4 \rightarrow \mathcal{M}_{\delta=\pi/2} = \begin{bmatrix} 0 & i/\gamma_1 \\ i\gamma_1 & 0 \end{bmatrix}$$

$$\lambda/2 \rightarrow \mathcal{M}_{\delta=\pi} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathcal{M}_{total} = \mathcal{M}_{\lambda/2}\,\mathcal{M}_{\lambda/4} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & i/\gamma_1 \\ i\gamma_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i/\gamma_1 \\ -i\gamma_1 & 0 \end{bmatrix}$$

eq (19-36) for $\mathcal{M}_{total}$

$$\Gamma = \frac{\gamma_0 M_{11} + \gamma_0\gamma_s M_{12} - M_{21} - \gamma_s M_{22}}{\gamma_0 M_{11} + \gamma_0\gamma_s M_{12} + M_{21} + \gamma_s M_{22}} = \frac{-i\gamma_0\gamma_s/\gamma_1 + i\gamma_1}{-i\gamma_0\gamma_s/\gamma_1 - i\gamma_1}$$

$$R_{total} = \Gamma^2 = \left(\frac{\gamma_0\gamma_s/\gamma_1 - \gamma_1}{\gamma_0\gamma_s/\gamma_1 + \gamma_1} \right)^2$$

for $\mathcal{M}_{\lambda/4}$

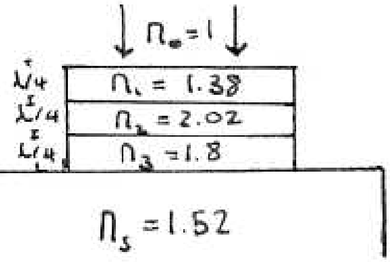
$$\Gamma = \frac{i\gamma_0\gamma_s/\gamma_1 - i\gamma_1}{i\gamma_0\gamma_s/\gamma_1 + i\gamma_1}$$

$$R_{\lambda/4} = \Gamma^2 = \left(\frac{\gamma_0\gamma_s/\gamma_1 - \gamma_1}{\gamma_0\gamma_s/\gamma_1 + \gamma_1} \right)^2$$

Thus,

$$\boxed{R_{total} = R_{\lambda/4}}$$

#3 (Pedrotti 19-11)



$$\mathcal{M}_{total} = \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3$$

$$= \begin{bmatrix} 0 & i/\gamma_1 \\ i\gamma_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & i/\gamma_2 \\ i\gamma_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & i/\gamma_3 \\ i\gamma_3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & i\gamma_1 \\ i\gamma_1 & 0 \end{bmatrix} \begin{bmatrix} -\gamma_3/\gamma_2 & 0 \\ 0 & -\gamma_2/\gamma_3 \end{bmatrix} = \begin{bmatrix} 0 & -i\gamma_2/\gamma_1\gamma_3 \\ -i\gamma_1\gamma_3/\gamma_2 & 0 \end{bmatrix}$$

From eq (19-36)

$$r = \frac{\gamma_0 M_{11} + \gamma_0 \gamma_S M_{12} - M_{21} - \gamma_S M_{22}}{\gamma_0 M_{11} + \gamma_0 \gamma_S M_{12} + M_{21} + \gamma_S M_{22}}$$

$$= \frac{\gamma_0 \gamma_S (-i\gamma_2/\gamma_1\gamma_3) + i\gamma_1\gamma_3/\gamma_2}{\gamma_0 \gamma_s (-i\gamma_2/\gamma_1\gamma_3) - i\gamma_1\gamma_3/\gamma_2}$$

$$= \frac{-\gamma_0\gamma_s\gamma_2^2 + \gamma_1^2\gamma_3^2}{-\gamma_0\gamma_s\gamma_2^2 - \gamma_1^2\gamma_3^2}$$

eq (19-12) $\gamma_i = n_i \sqrt{\epsilon_0 \mu_0} \cos\theta_{ti}$

light normal so $\theta_{ti} = 0$ $\quad = n_i \sqrt{\epsilon_0 \mu_0}$

$$= \frac{-\sqrt{\epsilon_0\mu_0}\, n_0\, n_s \sqrt{\epsilon_0\mu_0}\, n_2^2 \epsilon_0\mu_0 + n_1^2 \epsilon_0\mu_0\, n_3^2 \epsilon_0\mu_0}{-n_0\sqrt{\epsilon_0\mu_0}\, n_s \sqrt{\epsilon_0\mu_0}\, n_2^2 \epsilon_0\mu_0 - n_1^2 \epsilon_0\mu_0\, n_3^2 \epsilon_0\mu_0}$$

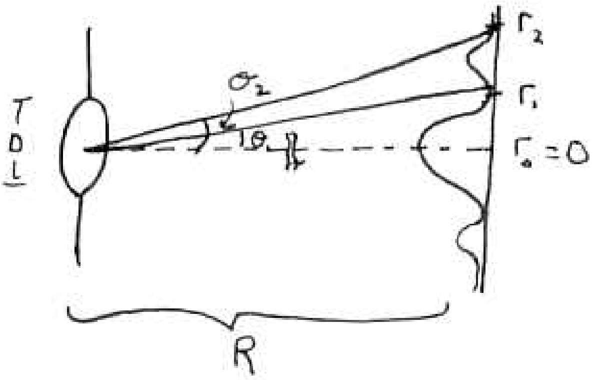
$$= \frac{-n_0 n_S n_2^2 + n_1^2 n_3^2}{-n_0 n_s n_2^2 - n_1^2 n_3^2}$$

We want zero reflectance $r = 0$

$$\Rightarrow n_1^2 n_3^2 = n_0 n_S n_2^2 \Rightarrow \left(\frac{n_1 n_3}{n_2}\right)^2 = n_0 n_s$$

$$\Rightarrow \boxed{\frac{n_1 n_3}{n_2} = \sqrt{n_0 n_s}}$$

#4 (Pedrotti 16-7)



$D = 36 \text{ in} = .9914\,m$

$f = 56\,ft$ = Distance from the aperature to screen

$= R = 17.07\,m$

$\gamma = \frac{kD}{2} \sin\theta \implies$ eq(16-20)

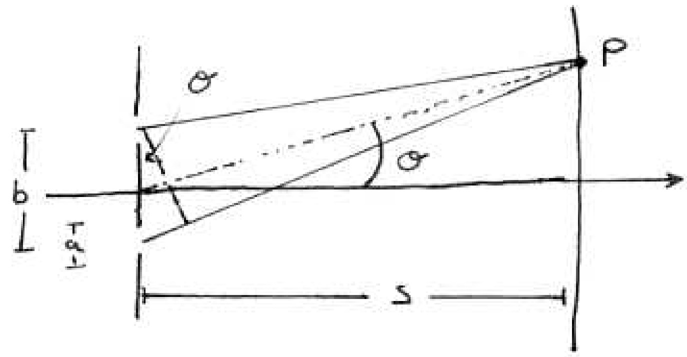
$\theta_1 = \frac{\gamma_1}{\pi} \frac{\lambda}{D} = \frac{5.14}{\pi} \frac{5.5 \times 10^{-7} m}{.9914\,m}$

$$\boxed{r_1 = R\theta_1 = 1.55 \times 10^{-5} m}$$

$r_2 = R\theta_2 = R \frac{\gamma_2}{\pi} \frac{\lambda}{D} = 17.07 \cdot \frac{8.42}{\pi} \cdot \frac{5.5 \times 10^{-7} m}{.9914\,m}$

$$\boxed{r_2 = 2.54 \times 10^{-5} m}$$

#5 (Pedrotti 16-13)



(a) Double Slit Deffraction Pattern:

$\lambda = 5.461 \times 10^{-7} m$

$b = 0.100\,mm = 1.00 \times 10^{-4}$

- Assuming that the fourth-order maximum is the first one missing from the pattern, which implies that this missing order is due to $m=1$ diffraction minimum:

eq (16-30) $a = \left(\frac{p}{m}\right) b = \frac{4}{1} \cdot 1.00 \times 10^{-4} m = \boxed{4.00 \times 10^{-4} m = \text{slit seperation}}$

(b) From eq. (16-27),

$$I = 4I_0 \left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha \quad ; \quad \begin{array}{l} \alpha = \tfrac{1}{2} ka\sin\theta \\ \beta = \tfrac{1}{2} kb\sin\theta \end{array}$$

• For the zeroth order max. $\theta \to 0$: $\cos^2\alpha \to 1$

and $\frac{\sin\beta}{\beta} \to \cos\beta \to 1$ (L'Hopital's Rule)

So, $\boxed{I_{0\,max} = 4I_0}$

• I_1 occurs at $p = 1$

eq. (16-29) $p\lambda = a\sin\theta$

$$\Rightarrow \sin\theta = \lambda/a \Rightarrow \alpha = \tfrac{1}{2}\frac{2\pi}{\lambda} a \frac{\lambda}{a} = \pi$$

$$\beta = \tfrac{1}{2}\frac{2\pi}{\lambda} b \cdot \lambda/a = \pi\frac{b}{a} = \frac{\pi}{4}$$

$$\Rightarrow I_1 = 4I_0\left(\frac{\sin\pi/4}{\pi/4}\right)^2 (-1)^2 = \frac{32}{\pi^2} = I_0$$

$$\Rightarrow \boxed{\frac{I_1}{I_0} = \frac{8}{\pi^2} = .811}$$

• I_2 at $p=2 \Rightarrow \sin\theta = \frac{2\lambda}{a} \Rightarrow \frac{I_2}{I_0} = \left(\frac{\sin\pi/2}{\pi/2}\right)^2 \Rightarrow \boxed{\frac{I_2}{I_0} = \frac{4}{\pi^2} = .405}$

• I_3 at $p=3 \Rightarrow \sin\theta = \frac{3\lambda}{a} \Rightarrow \frac{I_3}{I_0} = \left(\frac{\sin 3\pi/4}{3\pi/4}\right)^2$

$$\Rightarrow \boxed{\frac{I_3}{I_0} = \frac{8}{9}\frac{1}{\pi^2} = .090}$$

#6. (Pedrotti 16-22)

$$J_1(x) = \frac{\sin x - \cos x}{\sqrt{\pi x}} \quad \text{for large } x$$

$$\Rightarrow J_1(x) = 0 \quad \text{when} \quad \sin x = \cos x$$

$$\Rightarrow \text{Or } x_n = \frac{(2n+1)}{4}\pi \quad n = 0, 1, 2, \ldots$$

Now, from eq. (16-20)

$$① \; \gamma_n = \frac{kD}{2} \sin\theta_n = \left(\frac{2n+1}{4}\right)\pi$$

angle of diffraction min

$$② \; \gamma_{n+1} = \frac{kD}{2} \sin\theta_{n+1} = \left(\frac{2(n+1)+1}{4}\right)\pi \qquad \Delta\theta = \theta_{n+1} - \theta_n \ll 1$$

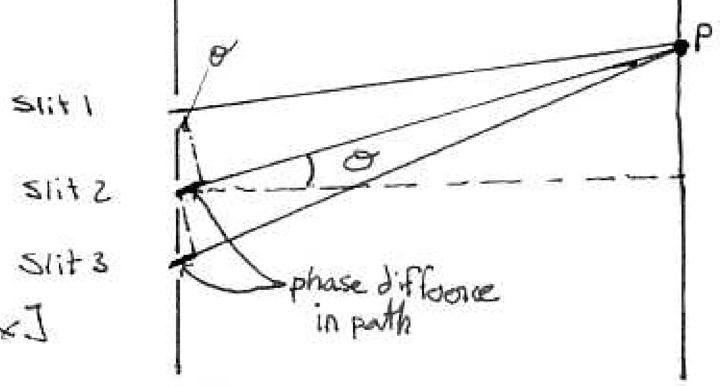
$$\Rightarrow \sin\theta_{n+1} = \sin(\theta_n + \Delta\theta) \approx \sin\theta_n + \Delta\theta\cos\theta_n$$

Plugging in eq ① and ② into the above,

$$\Rightarrow \frac{2(n+1)+1}{4}\pi = \frac{2n+1}{4}\pi + \Delta\theta\cos\theta_n \frac{kD}{2}$$

$$\Rightarrow \frac{\pi}{2} = \frac{kD}{2}\Delta\theta\cos\theta_n \Rightarrow \boxed{\Delta\theta = \frac{\lambda}{D\cos\theta}}$$

#7 (Pedrotti 16-25)



(a) $I_p = E_p^2 = 0$

$$\Rightarrow E_p = \tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3$$

[The tilde '~' indicated the fields are complex]

The middle slit creates a field equal to

$$\tilde{E}_2 = E_o$$

The other two slits will be equal amounts out of phase w/ the middle slit. E_1 will be ahead by φ, while E_2 will lag by φ_2:

$$\tilde{E}_1 = E_o e^{i\varphi} \quad \text{and} \quad \tilde{E}_3 = E_o e^{-i\varphi}$$

So,

$$E_p = E_o\left(1 + e^{i\varphi} + e^{-i\varphi}\right) = 0$$

$$\Rightarrow 2\cos\varphi = -1 \Rightarrow \boxed{\varphi = \frac{4\pi}{6} = 120^\circ}$$

(b) $\varphi = \pi \Rightarrow E_p = E_o\left(e^{i\pi} + 1 + e^{-i\pi}\right) = -E_o$

$$\Rightarrow I_p = E_p^2 = E_o^2 \qquad \Rightarrow \boxed{\frac{I_p}{I_o} = \frac{1}{9}}$$

[see page 344] $I_{max} = \underbrace{N^2}_{\text{number of slits}} I_o = 9 I_o$

(c) The first principal maximum is the central max ($p=0$)

$$\Rightarrow \boxed{I_p = I_{Max}}$$

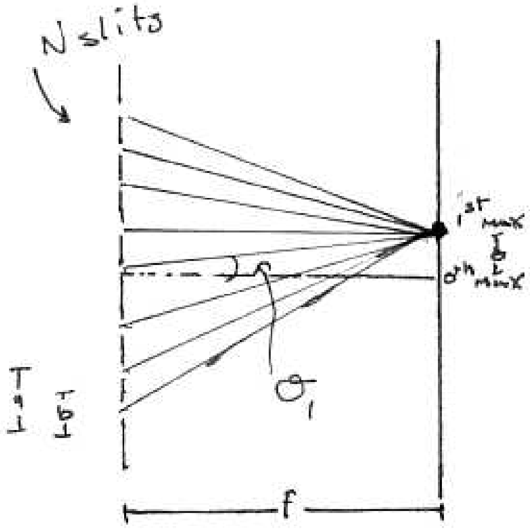
(d)

① Suppose we had light incident upon the three slits such that the light coming from them was mutually incoherent. In this case the irradiance pattern would be uniform.

② Now, if we make them coherent, then the maximum irradiance goes up by a factor of $\underline{\underline{3}}$, but the average stays the same.

i.e. $\boxed{\dfrac{I_{Max}}{I_{av}} = 3}$

8 (Pedrotti 17-8)



(1) $N=2$ (2) $N=10$ (3) $N=15{,}000$

$f=2m$; $a=5\times10^{-6}m$; $b=1\times10^{-6}m$

$\lambda=5.46\times10^{-7}m$

(a) Seperation between 0^{th} & 1^{st} order Max:

① This is dependent on the seperation between slits,
② but not the number of slits N.

Thus, the result is the same in all three cases:

eq(16-34) $m\lambda = a\sin\theta \Rightarrow$ $m=0 \rightarrow \theta_0=0$

$m=1 \rightarrow \theta_1 = \sin^{-1}\frac{\lambda}{a} = \sin^{-1}\left(\frac{5.46\times10^{-7}m}{5\times10^{-6}m}\right)$

$\theta_1 = .109$

$\boxed{\delta = f\theta_1 = (2m).109 = 21.8cm}$

(b) We have eq.(16-30)

condition for missing order $a = \left(\frac{p}{m}\right) b$

interference max. integer

diffraction min integer

central diffraction envelop $m=1$

thus, $P_{max} = \frac{a}{b} = \frac{.005mm}{.001mm} = 5 \Rightarrow$ So, there are $\boxed{\text{Nine}}$ bright fringes, for each number of slits $P=(0,\pm1,\pm2,\pm3,\pm4)$.

The last two $P_{max}=\pm5$, are not peaks

(c) The central fringe is centered at $\theta=0$ and ends at the first interference minimum, which is at

eq(16-32) $I = I_0\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2 = 0 \Rightarrow N\alpha = \pi$

$$\Rightarrow N \frac{ka}{2} \sin\theta = \pi$$

$$\sin\theta = \frac{2\lambda}{Na}$$

$$\theta_N = \sin^{-1}\left(\frac{2\lambda}{Na}\right)$$

Therefore,

$$W_N = f\, 2\theta_N \quad \Rightarrow \quad \boxed{\begin{aligned} W_2 &= 21.8\,\text{cm} \\ W_{10} &= 4.37\,\text{cm} \\ W_{10000} &= 2.9\times 10^{-5}\,\text{m} \end{aligned}}$$

University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

Problem Set 11

1. Pedrotti$\times$2 25-2.

2. Pedrotti$\times$2 25-5.

3. Pedrotti$\times$2 25-6.

4. Pedrotti$\times$2 25-7.

5. Pedrotti$\times$2 18-5.

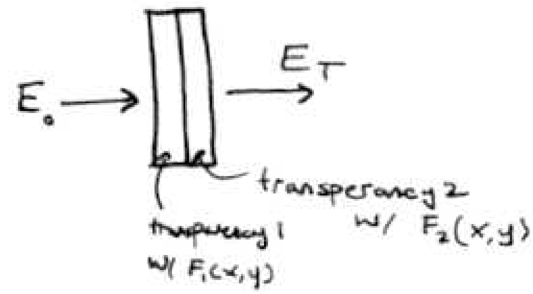
6. Pedrotti$\times$2 18-11.

7. Pedrotti$\times$2 18-17.

8. Pedrotti$\times$2 18-21.

Physics 110B
Homework #11

#1 (Pedrotti 25-2)



(a) consider the electric field of a light wave incident upon these two transperancies placed in series. After transversing the first transperancy the electric field transmitted is:

$$E_{T_1}(x,y) = F_1(x,y)\,E_0(x,y)$$

transmission function (e.g. a polarizer matrix, etc.)

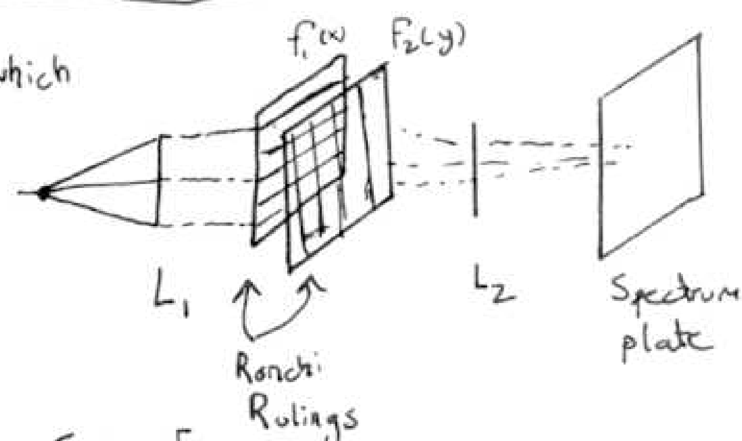
Now, this field goes through the next transperancy:

$$E_T(x,y) = F_2(x,y)\,E_{T_1}(x,y) = F_2(x,y)\,F_1(x,y)\,E_0(x,y)$$

$$\Rightarrow \boxed{F_{Total}(x,y) = \frac{E_T}{E_0} = F_2(x,y)\cdot F_1(x,y)}$$

So, we take the product of the two transmission func.

(b) Here we have two Ronchi rulings, which may be represented as:



(page 526) $f_1(x) = \frac{1}{2} + c\sum_{m=1}^{\infty}\frac{\cos mx}{m}$

$$f_2(\;) = 1/2 + c\sum_{n=1}^{\infty}\frac{\cos ny}{n}$$

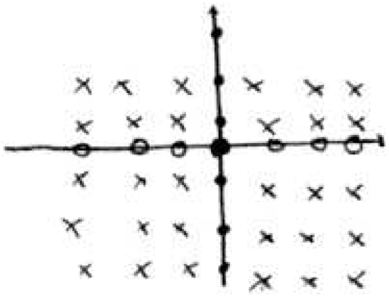
Using the above result, the total transmission Function F will be:

$$F(m,n) = \left(\frac{1}{2} + c\sum_{m=1}^{\infty}\frac{\cos mx}{m}\right)\left(\frac{1}{2} + c\sum_{n=1}^{\infty}\frac{\cos ny}{n}\right) = \text{Fourier transform of total transmission function}$$

$$= \frac{1}{4} + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\cos mx}{m} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos ny}{n} + \frac{4}{\pi^2}\sum_{m,n=1}^{\infty}\frac{\cos mx \cos ny}{nm}$$

DC — Ruling along y alone — Ruling along x alone — combinations of the two

Pattern on spectrum plate



x ~ "combinations"

o ~ x-alone

• ~ y-alone

#2 (Pedrotti 25-5)

$$h(x) = f(x) \otimes g(x) \Rightarrow h(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)\, e^{-iky}\, dk$$

$$g(x-y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\ell)\, e^{-i\ell(x-y)}\, d\ell$$

$$\Rightarrow h(x) = \frac{1}{(2\pi)^2}\int dy\, d\ell\, dk\, f(k)\, g(\ell)\, e^{-i\ell x} e^{-i(k-\ell)y}$$

$$\int e^{i(k-\ell)y} dy = 2\pi\,\delta(k-\ell)$$

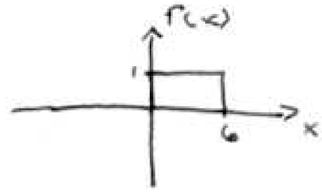
$$= \frac{1}{2\pi}\int dk\, f(k)\, g(k)\, e^{-ikx}$$

$$h(x) = \frac{1}{2\pi}\int h(k)\, e^{-ikx}\, dk \Rightarrow \boxed{h(k) = f(k)\, g(k)}$$

or $\boxed{\mathcal{F}(h(x)) = \mathcal{F}(f(x))\, \mathcal{F}(g(x))}$

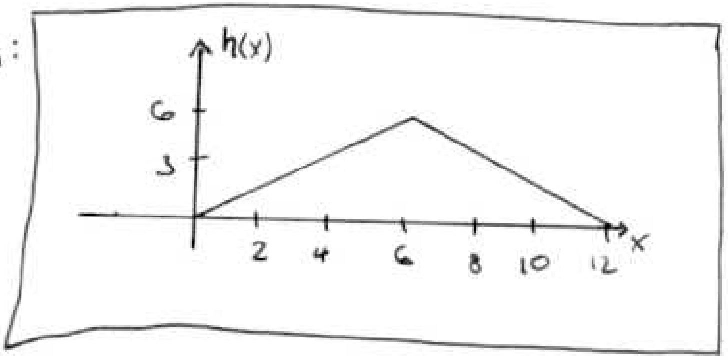
#3 (Pedrotti 25-6)

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 6 \\ 0, & \text{otherwise} \end{cases}$$



$$h(x) = f(x) \otimes f(x) = \int_{-\infty}^{\infty} f(y)\, f(x-y)\, dy = \int_0^6 f(x-y)\, dy$$

Which, when we graph:



#4 (Pedrotti 25-7)

$$f(t) = A \sin(\omega t + \alpha)$$

Using eq (25-30) in one-dimension

$$\Phi_{11}(\tau) = \int_0^{\tau} \big(A \sin(\omega(t+\tau) + \alpha)\big)\big(A \sin(\omega t + \alpha)\big)\, dt$$

$$= A^2 \int_0^{\tau} \sin(\omega t + \alpha)\big(\sin(\omega t + \alpha)\cos\omega\tau + \sin\omega\tau \cos(\omega t + \alpha)\big)\, dt$$

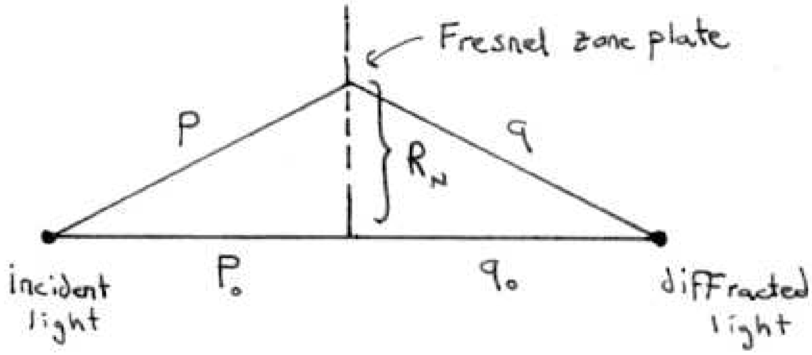
$$= A^2 \cos\omega\tau \int_{\alpha}^{2\pi+\alpha} \frac{1}{\omega} \sin^2 y\, dy = \frac{A^2}{\omega}\cos\omega\tau \left[\frac{y}{2} - \frac{\sin 2y}{4}\right]_{\alpha}^{2\pi+\alpha}$$

$$= \frac{A^2}{\omega}\cos\omega\tau \left(\pi + \frac{\alpha}{2} - \frac{\sin(4\pi + 2\alpha)}{4} - \frac{\alpha}{2} + \frac{\sin 2\alpha}{4}\right)$$

$$\boxed{\Phi_{11}(\tau) = \frac{\pi A^2}{\omega}\cos\omega\tau}$$

#5 (Pedrotti 18-5)

Using Figure (18-8)



For Fresnel zone, by definition the difference between the shortest path p_0+q_0 and the path to the n^{th} Fresnel zone is equal to $-N\lambda/2$:

$$① \quad (p_0+q_0) - (p+q) = -\frac{N\lambda}{2}$$

By the Pythagream theoren:

$$p^2 = p_0^2 + R_N^2 \qquad\qquad q^2 = q_0^2 + R_N^2$$

$$p = (p_0^2 + R_N^2)^{1/2} \qquad\qquad q = (q_0^2 + R_N^2)^{1/2}$$

$$= p_0\left(1 + \frac{R_N^2}{p_0^2}\right)^{1/2} \qquad\qquad = q_0\left(1 + \frac{R_N^2}{q_0^2}\right)^{1/2}$$

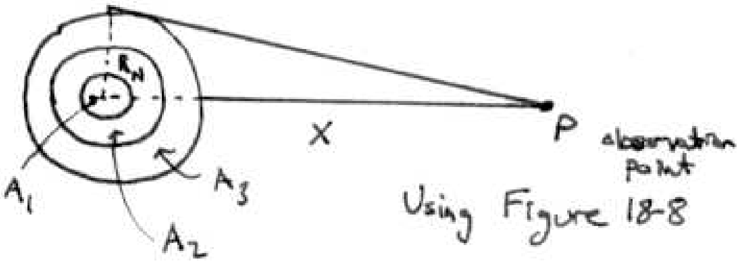
$$\simeq p_0 + \frac{R_N^2}{2p_0} \qquad \frac{R_N}{p_0} \ll 1 \quad \& \quad \frac{R_N}{q_0} \ll 1 \qquad \simeq q_0 + \frac{R_N^2}{2q_0}$$

Plugging these results into eq ①

$$\Rightarrow \frac{R_N^2}{2p_0} + \frac{R_N^2}{2q_0} = \frac{N\lambda}{2} \quad \Rightarrow \boxed{R_N = \sqrt{\frac{N\lambda\, p_0 q_0}{p_0+q_0}} = \sqrt{NL\lambda}}$$

#6 (Pedrotti 18-11)

The area of the N^{th} Fresnel half-period zone is,



Using Figure 18-8

$$\text{Area}_N = A_N = \pi R_N^2 - \pi R_{N-1}^2$$

using the result for plane waves eq (18-20):

$$R_N = \sqrt{N\lambda x} \implies A_N = \pi N\lambda x - \pi(N-1)\lambda x$$

$$\boxed{A_N = \pi\lambda x}$$

#7 (Pedrotti 18-17)

Using the Cornu spiral of Figure 18-12:

- The first max. occurs at $v = -1.2$
- The second max. occurs a full cycle 2π later at $v = -2.35$

From Table 18-1 (Fresnel Integrals)

$$v = -2.35 \implies \begin{matrix} C(v) = -.5908 \\ S(v) = -.5864 \end{matrix} \qquad H'$$

$$v = -\infty \implies \begin{matrix} C(-\infty) = -.5 \\ S(-\infty) = -.5 \end{matrix} \qquad E'$$

The magnitude of the phasor $H'E'$ is

$$\frac{E_p}{E_0} = \left((.5908 + .5)^2 + (.5864 + .5)^2 \right)^{1/2} = 1.54$$

$$\implies \boxed{I_p = (1.54)^2 E_0^2 = 2.37 I_0 = 1.19 I_u \quad \text{irradiance second max.}}$$

The next min. occurs half a cycle π later at $v=-2.75$

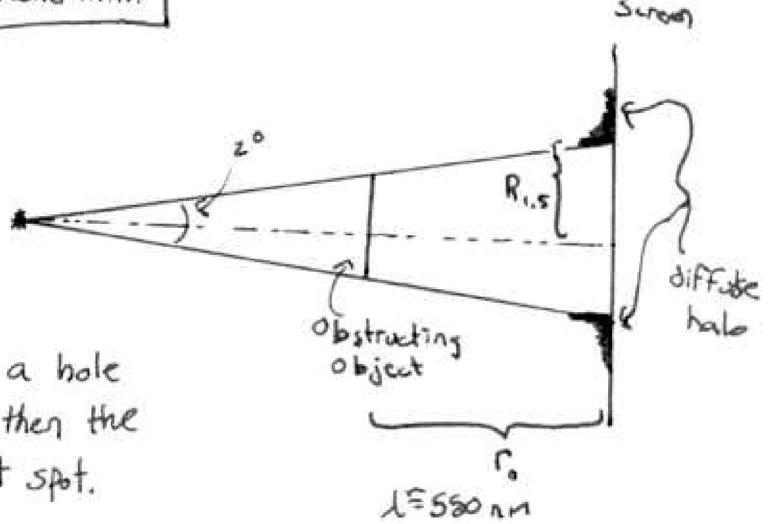
From Table 18-1

$$v=-2.75 \Rightarrow C(-2.75) = -.3908 \qquad J'$$
$$S(-2.75) = -.5015$$

Thus, the mag. of phasor $J'E'$ is,

$$E_P/E_o = \left((.3908+.5)^2 + (.5015+.5)^2\right)^{1/2} = 1.34$$

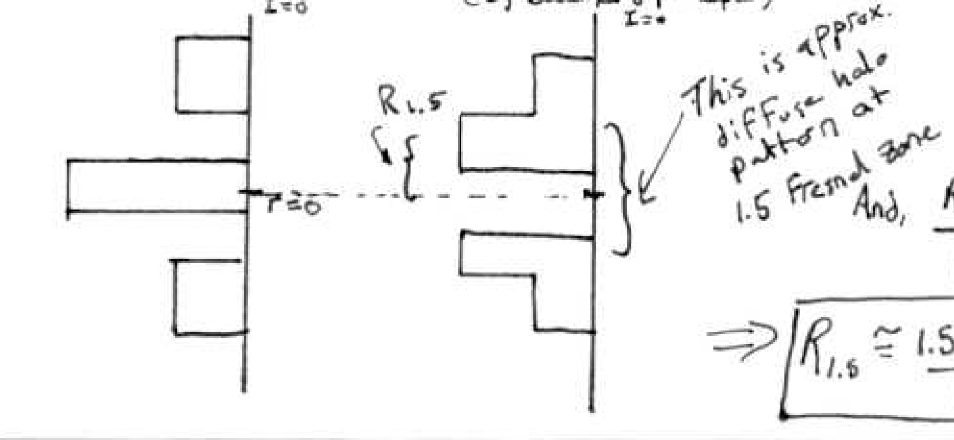
$$\Rightarrow \boxed{I_P = 1.797 I_o = .89 I_u \quad \text{irradiance second min.}}$$



8 (Pedrotti 18-21)

How many Fresnel zones should the particle cover in order to produce this halo?

- If we have an obstructing object with a hole which is one Fresnel zone wide, then then the pattern on the screen would be a bright spot.
- The complementary aperature, however, would just give a dark spot on the screen.
- However, covering slightly more then one Fresnel zone would give the correct complementary pattern of a halo.



So, Radius Particle $\approx R_{1.5}$

$R_{1.5}$ ~ radius between first and second Fresnel zone

$$R = R_{1.5} \approx \sqrt{1.5\lambda r_o} \quad \text{(eq 18-20)}$$

And, $\dfrac{R_{1.5}}{r_o} = \dfrac{2°}{2} = \dfrac{\pi}{180}$ radians

$$\Rightarrow \boxed{R_{1.5} \cong \frac{1.5\lambda \cdot 180}{2\pi} \cong 23.6 \text{ nm}}$$

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Problem Set 12

1. A simple hologram is made as follows: The object is a single narrow white strip located a distance z from the recording plate. The plate is illuminated at normal incidence by a reference laser beam of wavelength λ, which also illuminates the strip. Show that the resulting pattern on the hologram is a one-dimensional grating with a variable spacing s in the y direction, where y lies in the plane of the plate and is perpendicular to the strip. Give the numerical values of s for $z = 10$ cm and $\lambda = 633$ nm, for various values of y: 0, 1, 2, and 4 cm.

2. Referring to the previous problem, show in detail how, if the hologram is illuminated by the reference laser in the same way, two diffracted beams will emerge: one producing a real image of the strip, the other producing a virtual image. The second beam appears to diverge from a line corresponding to the original object, while the first converges toward a real image located symmetrically at $-z$ on the opposite side of the plate. Find the actual angles of diffraction for the various values of y given in the previous problem. Will there be second-order (or even higher-order) diffracted beams?

3. Pedrotti×2 13-1.

4. Pedrotti×2 13-12.

5. Pedrotti×2 22-1.

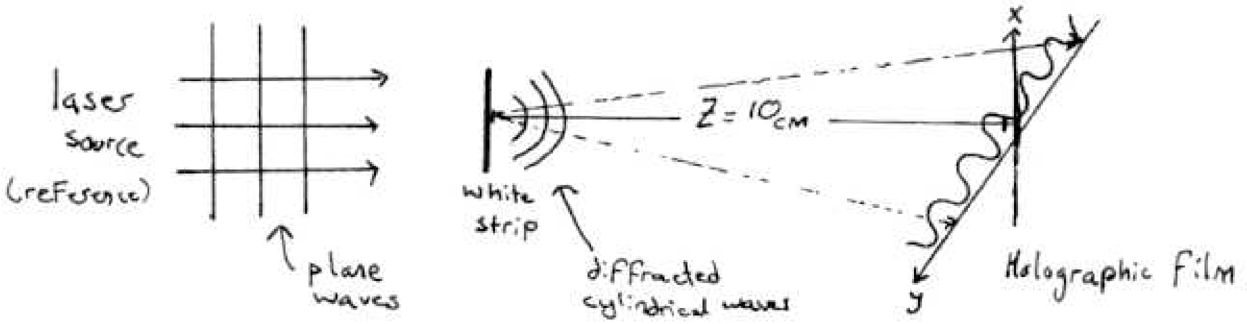
6. Pedrotti×2 22-11.

7. Pedrotti×2 22-15.

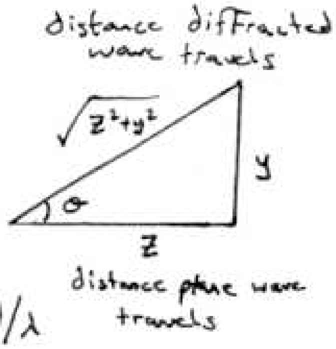
8. Pedrotti×2 22-17.

Physics 110B
Homework #12

#1.



- The stip is in the x-direction, therefore the reference beam will only be able to be diffracted about the sides of the strip. The diffracted subject wave will be cylinderical and will form a diffracting grating pattern in the y-direction on superposition with the reference beam.



- On the film there will be maximums when

$$\sqrt{y^2+z^2}-z = m\lambda \qquad m = 0, \pm 1, \pm 2, \ldots$$

$$\Rightarrow y_{max} = \left((m\lambda + z)^2 - z^2\right)^{1/2} \qquad \Rightarrow m = \left(\sqrt{y_m^2+z^2}-z\right)/\lambda$$

$$\Rightarrow y_{m+1} = \left(((m+1)\lambda + z)^2 - z^2\right)^{1/2}$$

$$= \left(\left(\sqrt{y_m^2+z^2}+\lambda\right)^2 - z^2\right)^{1/2}$$

$$= \left(y_m^2 + z^2 + \lambda^2 + 2\lambda\sqrt{y_m^2+z^2} - z^2\right)^{1/2}$$

$$= \left(y_m^2 + \lambda^2 + 2\lambda\sqrt{y_m^2+z^2}\right)^{1/2}$$

$$\Rightarrow y_{m+1} - y_m = \boxed{\left(y_m^2 + \lambda^2 + 2\lambda\sqrt{y_m^2+z^2}\right)^{1/2} - y_m = S}$$

$\lambda = 633 \times 10^{-9}\,m$
$z = .1\,m$

So,

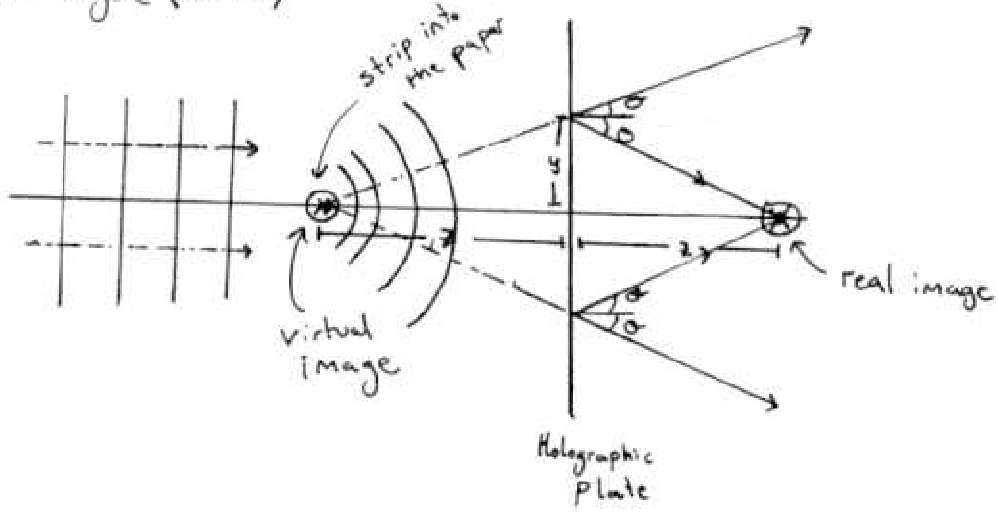
$$s = \left(y^2 + (633\times10^{-9})^2 + 2\cdot 633\times10^{-9}\sqrt{y^2 + .1^2}\right)^{1/2} - y$$

For

$$\boxed{\begin{array}{l} y = 0 \Rightarrow s = 3.56\times10^{-4}\,m \\ y = .01\,m \Rightarrow s = 6.36\times10^{-6}\,m \\ y = .02\,cm \Rightarrow s = 3.23\times10^{-6}\,m \\ y = .04\,cm \Rightarrow s = 1.70\times10^{-6}\,m \end{array}}$$

#2. The same argument Pedrotti gives on pages 267 - 269 for a point source will also be true for the thin strip.

In analogy to figure (13-1 b)



• When the plane wave hits the holographic plate it will diffract. At an angle of θ there will form constructively a real image on the positive z-side of the plate. This image is formed by the reference beam diffracted inwards, symmetrically the beam will defract outwards and form a virtual holographic image on the left side of the plate.

- In the case of a holographic plate, we have a semi-continuous zone pattern, therefore <u>there are no second or higher order diffracted beams.</u>

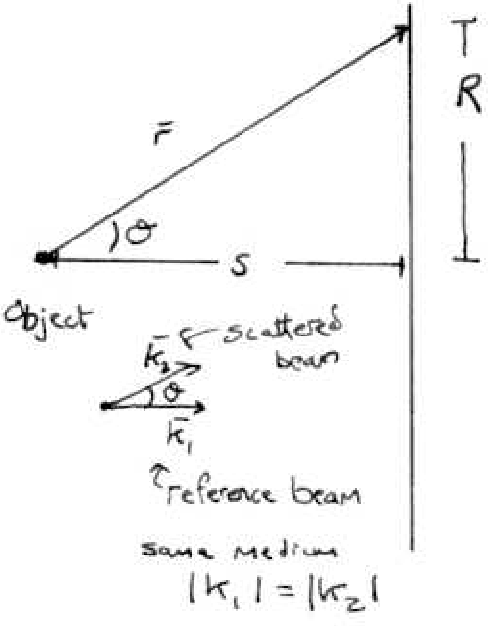
Now, from geometrical considerations the angle of diffraction for various heights y will be:

$$\Rightarrow \tan\theta = \frac{y}{z}$$

So,

$$\boxed{\begin{array}{lll} y = 1\,cm & \theta = \tan^{-1}\frac{1\,cm}{10\,cm} & = 5.71^\circ \\ y = 2\,cm & \theta = \tan^{-1}\frac{2\,cm}{10\,cm} & = 11.3^\circ \\ y = 4\,cm & \theta = \tan^{-1}\frac{4\,cm}{10\,cm} & = 21.8^\circ \end{array}}$$

#3 (Pedrotti 13-1)



From eq. (10.13)

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta,$$

$$\delta = (\bar{k}_1 - \bar{k}_2)\cdot\bar{r} + (\epsilon_1 - \epsilon_2)$$

There is no delay between interaction of incident beam with object & production of scattered wave:

$$\epsilon_1 - \epsilon_2 = 0$$

Also, $(\bar{k}_1 - \bar{k}_2)\cdot\bar{r} = kr\cos\theta - kr$

From Pythagorem's Theorem.

$$r = \sqrt{s^2 + R^2} = s\sqrt{1 + R^2/s^2} \approx s + R^2/2s$$

And, $\cos\theta = \dfrac{s}{\sqrt{s^2+R^2}} \approx 1 - \dfrac{R^2}{2s^2}$

$$\Rightarrow \delta = k\left(s + \frac{R^2}{2s}\right)\left(1 - R^2/2s^2\right) - k\left(s + R^2/2s\right)$$

$$= -\frac{kR^2}{2s} + O\left(\frac{R^4}{s^3}\right)$$

$$\Rightarrow \cos\delta \approx \cos 2aR^2; \quad a = \frac{\pi}{2s\lambda} = \frac{k}{4s}$$

$$\Rightarrow I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos 2aR^2 = I_1 + I_2 + 2\sqrt{I_1 I_2}\left(\cos^2 aR^2 - \sin^2 aR^2\right)$$

$$= I_1 + I_2 + 2\sqrt{I_1 I_2}\left(2\cos^2 aR^2 - 1\right) = \boxed{I_1 + I_2 - 2\sqrt{I_1 I_2} + 4\sqrt{I_1 I_2}\cos^2 aR^2}$$

#4 (Pedrotti 13-12)

(a) From the previous problem we have

$$I_F = A + B\cos^2 aR^2$$

which can be rewritten as.

$$= A + B\left(\frac{1+\cos 2aR^2}{2}\right) = \left(A+\frac{B}{2}\right) + \frac{B}{4}e^{2iaR^2} + \frac{B}{4}e^{-2iaR^2}$$

We now write down the equation for the plane wave reference beam. The object will be the origin $r=0$, thus

$$E_R = E_0 e^{i(\omega t - ks)}$$

s here because this gives the amplitude at the screen $r=s$, while $r=0$ is at the object

From eq (13-9)

$$\boxed{E_H \propto I_F E_R = \left(\left(A+\frac{B}{2}\right) + \frac{B}{4}e^{ikR^2/2s} + \frac{B}{4}e^{-ikR^2/2s}\right)E_0 e^{i(\omega t - ks)}}$$

① ② ③

The subject beam has the form of a spherical wave,

$$E_s = \frac{D}{r}e^{i(\omega t - kr)} \approx \frac{D}{s}\left(1-\frac{R^2}{2s^2}\right)e^{i(\omega t - ks - kR^2/2s)}$$

$$= D' e^{i(\omega t - ks - kR^2/(2s))}$$

Therefore, the first term of the boxed equation corresponds to

① $\boxed{E_{H1} = \left(A+\frac{B}{2}\right)e^{i(\omega t - ks)}}$ • the reference beam modulated in amplitude but not in phase

The second term ② corresponds to

② $$\boxed{E_{H2} = \frac{B}{4} e^{i(\omega t - ks + kR^2/2s)}}$$

- the subject beam modulated in amplitude and has a phase reversal (+ sign in front of $kR^2/2s$)
- Produces phase reversed real image

The third term ③ corresponds to

③ $$\boxed{E_{H3} = \frac{B}{4} e^{i(\omega t - ks - kR^2/2s)}}$$

- the virtual image with correct phase

(b) From the previous problem we have

$$r \approx s + \frac{R^2}{2s}$$

$$\Rightarrow \delta = k(r-s) \approx \frac{2\pi}{\lambda}\left(\frac{R^2}{2s}\right) = \boxed{\frac{\pi R^2}{\lambda s} = \delta}$$

For the subject beam we have

$$E_s = D' e^{i(\omega t - ks - \underbrace{kR^2/2s}_{\delta})}$$

If the sign of the phase is reversed we have,

$$E_s = D' e^{i(\omega t - ks + kR^2/2s)}$$

which is just E_{H2} which is a converging real image

#5 (Pedrotti 22-1)

(a) From eq. (22-6)

$$r = R\left(1 + \frac{x^2+y^2}{R^2}\right)^{1/2}$$

$$\boxed{r \approx R + \frac{x^2+y^2}{2R}} \Leftarrow \frac{x^2+y^2}{R^2} \ll 1$$

Thus, from eq. (22-5)

(b) $$\Rightarrow \boxed{\tilde{E}(x, y, z=R) \cong C e^{ikR} e^{ik(x^2+y^2)/2R}}$$

#6 (Pedrotti 22-11)

$$\frac{\Phi(r=a)}{\Phi_{tot}} = \frac{1}{\Phi_{tot}} \iint_{\text{aperature}} \left|\tilde{E}(x,y,z,t)\right|^2 dA$$

using eq. (22-18)

$$= \frac{1}{\Phi_{tot}} \iint_A \left| E_0 e^{ik(x^2+y^2)/2R(z)} e^{-(x^2+y^2)/w^2(z)} e^{i(kz+p(z)-\omega t)}\right|^2 dA$$

$$= \frac{E_0^2}{\Phi_{tot}} \iint_A e^{-2(x^2+y^2)/w^2(z)} dA$$

$$= \frac{E_0^2}{\Phi_{tot}} \int_0^a r\,dr \int_0^{2\pi} d\varphi\, e^{-2r^2/w^2(z)}$$

Φ_{tot} (from page 473)

$$= E_0^2 \cdot \frac{2}{E_0^2 \pi w^2(z)} \cdot 2\pi \cdot \frac{w^2(z)}{4} \cdot \left(1 - e^{-2a^2/w^2(z)}\right)$$

$$\Rightarrow \boxed{\frac{\Phi(r=a)}{\Phi_{tot}} = 1 - e^{-2a^2/w^2(z)}}$$

#7 (Pedrotti 22-15)

From eq. (22-58)

$$H_m(\xi) = (-1)^m e^{\xi^2} \frac{d^m}{d\xi^m}\left(e^{-\xi^2}\right)$$

$$m=0 \Rightarrow H_0(\xi) = e^{\xi^2} \cdot e^{-\xi^2} = 1 \checkmark$$

$$m=1 \Rightarrow H_1(\xi) = (-1) e^{\xi^2} \frac{d}{d\xi} e^{-\xi^2} = 2\xi \checkmark$$

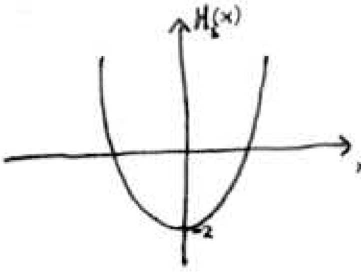
$$m=2 \Rightarrow H_2(\xi) = e^{\xi^2} \frac{d^2}{d\xi^2}\left(e^{-\xi^2}\right) = e^{\xi^2}\left(4\xi^2 e^{-\xi^2} - 2e^{-\xi^2}\right)$$

$$= 4\xi^2 - 2 \checkmark$$

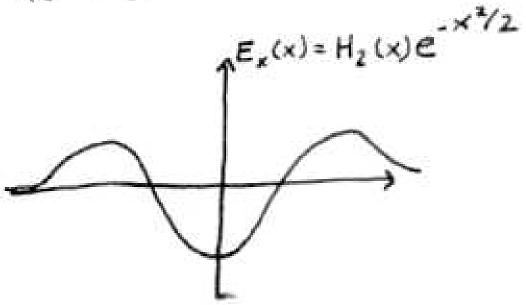
#8 (Pedrotti 22-17)

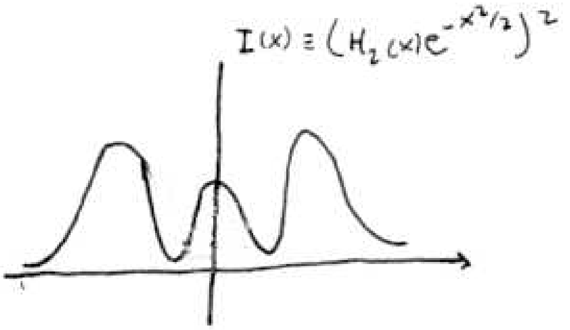
Following Figure 22-17

$$H_{m=2}(x) = \frac{8x^2}{W^2} - 2 \Rightarrow E \propto \left(\frac{8x^2}{W} - 2\right) e^{-(x^2+y^2)/2}$$



$\Rightarrow$





Burn Pattern:

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MIDTERM EXAMINATION

Directions. Do all three problems, which have unequal weight. This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. Calculators are not needed; where numerical results are requested, 30% accuracy is sufficient. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

Problem 1. (35 points) A coaxial transmission line consists of two perfectly conducting circular cylindrical thin-walled tubes of radii a and b, respectively, both centered on the $\hat{\mathbf{z}}$ axis. The region $a < r < b$ is evacuated. Consider propagation of electromagnetic waves in the TEM mode ($E_z = B_z = 0$) within the vacuum region. Take

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \Re(\tilde{\mathbf{E}}(r,\phi)e^{i(kz-\omega t)}) \\ \mathbf{B}(\mathbf{r},t) &= \Re(\tilde{\mathbf{B}}(r,\phi)e^{i(kz-\omega t)})\ ,\end{aligned}$$

where $k = \omega/c$. Then, in the vacuum region, Maxwell's equations reduce to

$$\nabla_t \cdot \tilde{\mathbf{E}} = \nabla_t \times \tilde{\mathbf{E}} = 0\ ,$$

and

$$c\tilde{\mathbf{B}} = \hat{\mathbf{z}} \times \tilde{\mathbf{E}}\ ,$$

where

$$\nabla_t \equiv \nabla - \hat{\mathbf{z}}\frac{\partial}{\partial z}\ .$$

(a) (5 points) Show that $\tilde{\mathbf{E}}$ can be written as

$$\tilde{\mathbf{E}}(r,\phi) = -\nabla_t\tilde{\Phi}(r,\phi)$$

where

$$\nabla_t^2\tilde{\Phi} = 0\ .$$

(If you don't manage to show this, you may nevertheless assume this result in the later parts.)

(b) (15 points) Assume that $\tilde{\Phi} = \Phi_0$ on the outer cylinder, and $\tilde{\Phi} = 0$ on the inner cylinder, where Φ_0 is a real constant. Calculate the physical (real) fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ in the gap between the cylinders, in terms of Φ_0.

(c) (15 points) Find Z_0, the characteristic impedance of this transmission line. Z_0 may be defined as the ratio of Φ_0 to the maximum total current flowing on either cylindrical surface. Assume that this current is distributed uniformly in ϕ. Evaluate Z_0 in ohms for the case $b/a = 2.71828$.

Problem 2. (30 points) Event A happens at spacetime point $(ct,x,y,z) = (2,0,0,0)$; event B occurs at (0,1,1,1), both in an inertial system S.

(a) (10 points) Is there an inertial system S' in which events A and B occur at the same spatial coordinates? If so, find $c|t'_A - t'_B|$, c times the magnitude of the time interval in S' between the two events.

(b) (10 points) Is there an inertial system S'' in which events A and B occur simultaneously? If so, find $|\mathbf{r}''_A - \mathbf{r}''_B|$, the distance in S'' between the two events.

(c) (10 points) Can event A be the cause of event B, or vice versa? Explain.

Problem 3. (35 points) A point charge e travelling on the x axis has position

$$\begin{aligned}\mathbf{r}(t) &= +\hat{\mathbf{x}}\frac{ct}{2} \quad (t < 0)\\ &= -\hat{\mathbf{x}}\frac{ct}{2} \quad (t > 0)\ .\end{aligned}$$

That is, the charge reverses direction instantaneously at $t = 0$, while it is at the origin. The fields that the charge produces are viewed by an observer at $(x, y, z) = (0, 1, 0)$ m.

(a) (20 points) What magnetic field $\mathbf{B}$ does the observer see at $t = 0$?

(b) (15 points) At time t such that $ct = 1$ m, what is the direction of the electric field $\mathbf{E}$ seen by the observer? (You need consider only the part of the total electric field which is dominant at exactly that time.) Justify your answer.

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MIDTERM EXAMINATION

Directions: Do all three problems, which have unequal weight. This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. Calculators are not needed, but you may use one if you wish. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

Problem 1. (35 points) A parallel-strip transmission line consists of two perfectly conducting long flat thin metal strips of width D and vacuum separation $d \ll D$ extending a long distance in the $\hat{\mathbf{z}}$ direction. Take $\hat{\mathbf{x}}$ to be normal to the strips, and define $x = y = 0$ at the midpoint of the gap. Consider the propagation in the gap of electromagnetic waves of angular frequency ω in the TEM mode ($E_z = B_z = 0$). Take

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\big(\tilde{\mathbf{E}}(x, y)e^{i(\kappa z - \omega t)}\big)$$
$$\mathbf{B}(\mathbf{r}, t) = \mathrm{Re}\big(\tilde{\mathbf{B}}(x, y)e^{i(\kappa z - \omega t)}\big) \ ,$$

where $\kappa \equiv \omega/c$. Then, in vacuum, the relevant Maxwell equations reduce to

$$0 = \frac{\partial \tilde{E}_x}{\partial x} + \frac{\partial \tilde{E}_y}{\partial y}$$
$$0 = \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y}$$
$$c\tilde{B}_y = \tilde{E}_x$$
$$-c\tilde{B}_x = \tilde{E}_y \ .$$

(**a**) (10 points) Show that $\tilde{\mathbf{E}}$ can be written as

$$\tilde{\mathbf{E}}(\mathbf{x}, \mathbf{y}) = -\nabla_t \tilde{\Phi}(x, y) \ ,$$

where

$$\nabla_t^2 \tilde{\Phi} = 0$$

and

$$\nabla_t \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} \ .$$

If you don't manage to show this, you nevertheless should assume this result in the later parts.
(**b**) (10 points) Assume that

$$\tilde{\Phi} = +\Phi_0/2$$

on the top plate, and

$$\tilde{\Phi} = -\Phi_0/2$$

on the bottom plate, where Φ_0 is a real constant. Neglecting the small region near the edges, calculate the *real physical* fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ in the gap between the plates, in terms of Φ_0.
(**c**) (15 points) Find the characteristic impedance Z of this transmission line. Evaluate Z in ohms for the case $D = 100d$.
[*Hint*: One way to do this is to take Z to be the ratio of Φ_0 to the maximum current flowing on the inner surface of either plate. Assume that this current is distributed uniformly in y. Another way is to take Z to be $\sqrt{L/C}$, where L and C are the inductance and capacitance per unit length of the transmission line, and $\sqrt{1/LC}$ is the phase velocity of the wave.]

Problem 2. (40 points) "Surface" muon beams are important tools for investigating the properties of condensed matter samples as well as fundamental particles. Protons from a cyclotron produce π^+ mesons (quark-antiquark pairs) that come to rest near the surface of a solid target. The pion then decays isotropically to an (anti)muon (μ^+) and a neutrino (ν) via

$$\pi^+ \to \mu^+ + \nu \ .$$

Some of the muons can be captured by a beam channel and transported in vacuum to an experiment. In the limit that the mother pion decays at the surface of the target (so that the daughter muon traverses negligible material), the beam muons have uniform speed (and, as it turns out, 100% polarization as well). For the purposes of this problem, consider a muon to have 3/4 of the rest mass of a pion; neglect the neutrino mass.

(**a**) (15 points) Show that the surface muons travel at a speed which is a fraction $\beta_0 = 0.28$ of the speed of light.

(**b**) (15 points) A good method for capturing and transporting surface muons is to place the muon production target on the axis of a solenoidal magnet with uniform field B; this axis defines the beam direction. Muons (of charge e and rest mass m) that are emitted close to the axial direction are captured and transported by the solenoid. In terms of β_0 and other constants, over what path length L does a surface muon travel before it returns to the solenoid axis?

(**c**) (10 points) If a muon's mean proper lifetime is τ, what fraction of the muons will decay before they return to the solenoid axis? (Just in case you didn't get part (**a**) or (**b**) quite right, please leave your answer in terms of β_0 and L.)

Problem 3. (25 points) Consider the interaction of an electron of charge $-e$ and mass m with an (externally produced) electromagnetic field described by the four-potential A^μ. The interaction Lagrangian L_{int} in this case is

$$L_{\text{int}} = -\frac{e}{\gamma m} p_\mu A^\mu \ ,$$

where p^μ is the particle's four-momentum. Consider the canonical momentum

$$P^\mu \equiv p^\mu - eA^\mu \ .$$

If one applies the Euler-Lagrange equations to L_{int}, one discovers that if all four components of A^μ are independent of any spatial coordinate x^i, then P^i, the i^{th} component of P^μ, is *conserved.*

While these facts may seem like theoretical niceties, they can be of practical use. Consider a capacitor whose parallel plates lie in the xy plane. The inside of the bottom plate is at $z = 0$ and the inside of the top plate is at $z = d$. The bottom plate is grounded, and a positive voltage V_0 is applied to the top plate. The whole setup is bathed in a uniform magnetic field

$$\mathbf{B} = \hat{\mathbf{y}} B_0 \ ,$$

which can be derived from a vector potential

$$\mathbf{A} = \hat{\mathbf{x}} B_0 z \ .$$

An electron is emitted from the bottom plate in the z direction with negligible velocity. It is accelerated in the z direction toward the top plate by the electric field in the gap; however, as the electron gains velocity, the Lorentz force from the magnetic field bends it toward the x direction. The resulting motion is complicated.

(**a**) (15 points) Show that the x component of the electron's momentum varies only as a function of its altitude z, and find the dependence.

(**b**) (10 points) For simplicity assuming that the electron is nonrelativistic, and taking B_0 to be fixed, find the minimum value of the applied voltage V_0 such that the electron makes it all the way up to the top plate.

[The above describes an oversimplified version of the *static magnetron tube*, which generated the radar signals that won the Battle of Britain.]

University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

SOLUTION TO MIDTERM EXAMINATION

Directions: Do all three problems, which have unequal weight. This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. Calculators are not needed, but you may use one if you wish. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

Problem 1. (35 points) A parallel-strip transmission line consists of two perfectly conducting long flat thin metal strips of width D and vacuum separation $d \ll D$ extending a long distance in the $\hat{\mathbf{z}}$ direction. Take $\hat{\mathbf{x}}$ to be normal to the strips, and define $x = y = 0$ at the midpoint of the gap. Consider the propagation in the gap of electromagnetic waves of angular frequency ω in the TEM mode ($E_z = B_z = 0$). Take

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= \mathrm{Re}\big(\tilde{\mathbf{E}}(x, y)e^{i(\kappa z-\omega t)}\big)\\ \mathbf{B}(\mathbf{r}, t) &= \mathrm{Re}\big(\tilde{\mathbf{B}}(x, y)e^{i(\kappa z-\omega t)}\big)\ ,\end{aligned}$$

where $\kappa \equiv \omega/c$. Then, in vacuum, the relevant Maxwell equations reduce to

$$\begin{aligned}0 &= \frac{\partial \tilde{E}_x}{\partial x} + \frac{\partial \tilde{E}_y}{\partial y}\\ 0 &= \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y}\\ c\tilde{B}_y &= \tilde{E}_x\\ -c\tilde{B}_x &= \tilde{E}_y\ .\end{aligned}$$

(**a**) (10 points) Show that $\tilde{\mathbf{E}}$ can be written as

$$\tilde{\mathbf{E}}(\mathbf{x}, \mathbf{y}) = -\nabla_t \tilde{\Phi}(x, y)\ ,$$

where

$$\nabla_t^2 \tilde{\Phi} = 0$$

and

$$\nabla_t \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y}\ .$$

If you don't manage to show this, you nevertheless should assume this result in the later parts.

Solution:

$$\begin{aligned}\nabla \times \tilde{\mathbf{E}} &= \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} = 0\\ \Rightarrow \tilde{\mathbf{E}} &= -\nabla\tilde{\Phi} = -\nabla_t\tilde{\Phi}\\ 0 &= \frac{\partial \tilde{E}_x}{\partial x} + \frac{\partial \tilde{E}_y}{\partial y}\\ &= \nabla_t \cdot \tilde{\mathbf{E}}\\ &= \nabla_t \cdot (-\nabla_t\tilde{\Phi})\\ &= -\nabla_t^2\tilde{\Phi}\ .\end{aligned}$$

(**b**) (10 points) Assume that

$$\tilde{\Phi} = +\Phi_0/2$$

on the top plate, and

$$\tilde{\Phi} = -\Phi_0/2$$

on the bottom plate, where Φ_0 is a real constant. Neglecting the small region near the edges, calculate the *real physical* fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ in the gap between the plates, in terms of Φ_0.

Solution:

The solution $\tilde{\Phi}$ to Laplace's equation is unique, given the boundary conditions:

$$\tilde{\Phi} = \Phi_0\frac{x}{d}\ .$$

Therefore

$$\begin{aligned}\tilde{\mathbf{E}} &= -\hat{\mathbf{x}}\frac{\Phi_0}{d}\\ \mathbf{E} &= -\hat{\mathbf{x}}\frac{\Phi_0}{d}\cos(\kappa z-\omega t)\\ c\mathbf{B} &= -\hat{\mathbf{y}}\frac{\Phi_0}{d}\cos(\kappa z-\omega t)\ .\end{aligned}$$

(**c**) (15 points) Find the characteristic impedance Z of this transmission line. Evaluate Z in ohms for the case $D = 100d$.
[*Hint*: One way to do this is to take Z to be the ratio of Φ_0 to the maximum current flowing on the inner surface of either plate. Assume that this current is distributed uniformly in y. Another way is to take Z to be $\sqrt{L/C}$, where L and C are the inductance and capacitance per unit length of the transmission line, and $\sqrt{1/LC}$ is the phase velocity of the wave.]
Solution:
Method 1:
There are no time-varying fields within the perfect conductors. Therefore, across the inner boundary of either conductor, from Ampère's law, ignoring directions and signs,

$$\begin{aligned} \Delta B &= \mu_0 K \\ B_{\max} &= \mu_0 K_{\max} \ , \end{aligned}$$

where $K_{\max}$ is the maximum surface current density (amperes/m). The impedance is

$$\begin{aligned} Z &= \frac{\Phi_0}{K_{\max} D} \\ &= \frac{\mu_0 \Phi_0}{B_{\max} D} \\ &= \frac{c\mu_0 \Phi_0}{E_{\max} D} \\ &= \frac{c\mu_0 \Phi_0 d}{\Phi_0 D} \\ &= \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{d}{D} \\ &= 3.77\ \Omega \ . \end{aligned}$$

Method 2:

$$\begin{aligned} v_{\text{ph}} &= \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \frac{1}{\sqrt{LC}} \\ L &= \frac{\epsilon_0 \mu_0}{C} \\ C &= \frac{\epsilon_0 D}{d} \\ Z &= \sqrt{\frac{L}{C}} = \sqrt{\frac{\epsilon_0 \mu_0}{C^2}} \\ &= \sqrt{\frac{\epsilon_0 \mu_0 d^2}{\epsilon_0^2 D^2}} \\ &= \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{d}{D} \\ &= 3.77\ \Omega \ . \end{aligned}$$

Problem 2. (40 points) "Surface" muon beams are important tools for investigating the properties of condensed matter samples as well as fundamental particles. Protons from a cyclotron produce π^+ mesons (quark-antiquark pairs) that come to rest near the surface of a solid target. The pion then decays isotropically to an (anti)muon (μ^+) and a neutrino (ν) via

$$\pi^+ \to \mu^+ + \nu \ .$$

Some of the muons can be captured by a beam channel and transported in vacuum to an experiment. In the limit that the mother pion decays at the surface of the target (so that the daughter muon traverses negligible material), the beam muons have uniform speed (and, as it turns out, 100% polarization as well). For the purposes of this problem, consider a muon to have 3/4 of the rest mass of a pion; neglect the neutrino mass.
(**a**) (15 points) Show that the surface muons travel at a speed which is a fraction $\beta_0 = 0.28$ of the speed of light.
Solution:
Let μ, π, and ν be the four-momenta of the muon, pion, and neutrino, respectively, with units such that $c = 1$. Enforcing energy-

momentum conservation,

$$\begin{aligned}
\pi &= \mu + \nu \\
\nu &= \pi - \mu \\
\nu \cdot \nu &= (\pi - \mu) \cdot (\pi - \mu) \\
0 &= \pi \cdot \pi + \mu \cdot \mu - 2\pi \cdot \mu \\
0 &= m_\pi^2 + m_\mu^2 - 2m_\pi E_\mu \\
E_\mu &= \frac{m_\pi^2 + m_\mu^2}{2m_\pi} \; .
\end{aligned}$$

Similarly, permuting the same equation, and using $E_\nu = p_\nu = p_\mu$,

$$\begin{aligned}
\mu &= \pi - \nu \\
\mu \cdot \mu &= (\pi - \nu) \cdot (\pi - \nu) \\
m_\mu^2 &= \pi \cdot \pi + \nu \cdot \nu - 2\pi \cdot \nu \\
&= m_\pi^2 + 0 - 2m_\pi E_\nu \\
&= m_\pi^2 - 2m_\pi p_\mu \\
p_\mu &= \frac{m_\pi^2 - m_\mu^2}{2m_\pi} \; .
\end{aligned}$$

Taking the ratio of these two results

$$\begin{aligned}
\beta_0 &= \frac{p_\mu}{E_\mu} \\
&= \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} \\
&= \frac{\frac{16}{9} - 1}{\frac{16}{9} + 1} = \frac{7}{25} = 0.28 \; .
\end{aligned}$$

(**b**) (15 points) A good method for capturing and transporting surface muons is to place the muon production target on the axis of a solenoidal magnet with uniform field B; this axis defines the beam direction. Muons (of charge e and rest mass m) that are emitted close to the axial direction are captured and transported by the solenoid. In terms of β_0 and other constants, over what path length L does a surface muon travel before it returns to the solenoid axis?

Solution:

The motion is helical with angular frequency equal to the (relativistic) cyclotron frequency. Working in the lab,

$$\begin{aligned}
\Omega_{\text{cyclotron}} &= \frac{eB}{\gamma m} \\
&= \frac{eB\sqrt{1-\beta_0^2}}{m} \\
T &= \frac{2\pi}{\Omega_{\text{cyclotron}}} \\
&= \frac{2\pi m}{eB\sqrt{1-\beta_0^2}} \\
L &= \beta_0 c T \\
&= \frac{2\pi m \beta_0 c}{eB\sqrt{1-\beta_0^2}} \; .
\end{aligned}$$

(**c**) (10 points) If a muon's mean proper lifetime is τ, what fraction of the muons will decay before they return to the solenoid axis? (If you are concerned that you didn't get part (**a**) or (**b**) quite right, you may leave your answer in terms of β_0 and L.)

Solution:

In the lab, the time interval before the muon returns to the solenoid axis is $T = L/(\beta_0 c)$ (above). In the proper (rest) frame of the muon, the same interval is $T' = T/\gamma_0$. If the mean life is τ, the survival probability at time T' is $\exp(-T'/\tau)$. Therefore the fraction F of muons that fail to survive before returning to the solenoid axis is

$$\begin{aligned}
F &= 1 - \exp(-T'/\tau) \\
&= 1 - \exp\big(-T/(\gamma_0 \tau)\big) \\
&= 1 - \exp\left(-\frac{L}{c\tau}\frac{\sqrt{1-\beta_0^2}}{\beta_0}\right) \; .
\end{aligned}$$

The above is an acceptable solution. Expressed in terms of the answer to (**b**), it is

$$F = 1 - \exp\left(-\frac{2\pi m}{eB\tau}\right) \; ,$$

independent of β_0.

Problem 3. (25 points) Consider the interaction of an electron of charge $-e$ and mass m with an (externally produced) electromagnetic field described by the four-potential A^μ. The interaction Lagrangian L_{int} in this case is

$$L_{\text{int}} = -\frac{e}{\gamma m} p_\mu A^\mu \; ,$$

where p^μ is the particle's four-momentum. Consider the canonical momentum

$$P^\mu \equiv p^\mu - eA^\mu \ .$$

If one applies the Euler-Lagrange equations to L_{int}, one discovers that if all four components of A^μ are independent of any spatial coordinate x^i, then P^i, the i^{th} component of P^μ, is *conserved*.

While these facts may seem like theoretical niceties, they can be of practical use. Consider a capacitor whose parallel plates lie in the xy plane. The inside of the bottom plate is at $z = 0$ and the inside of the top plate is at $z = d$. The bottom plate is grounded, and a positive voltage V_0 is applied to the top plate. The whole setup is bathed in a uniform magnetic field

$$\mathbf{B} = \hat{\mathbf{y}} B_0 \ ,$$

which can be derived from a vector potential

$$\mathbf{A} = \hat{\mathbf{x}} B_0 z \ .$$

An electron is emitted from the bottom plate in the z direction with negligible velocity. It is accelerated in the z direction toward the top plate by the electric field in the gap; however, as the electron gains velocity, the Lorentz force from the magnetic field bends it toward the x direction. The resulting motion is complicated.

(**a**) (15 points) Show that the x component of the electron's momentum varies only as a function of its altitude z, and find the dependence.

Solution:

The components of A^μ are

$$\begin{aligned} A^0 &= \frac{V}{c} = \frac{V_0 z}{cd} \\ \mathbf{A} &= \hat{\mathbf{x}} B_0 z \ . \end{aligned}$$

Each component of A^μ is independent of both x and y. Therefore, both the x and y components of P^μ are conserved. Since $\mathbf{A}$ has no y component, conservation of the y component of P^μ merely confirms that the electron moves in the xz plane, which we could have deduced from the Lorentz force law. In the x direction,

$$\begin{aligned} p_x - eA_x &= \big(p_x(z=0) - eA_x(z=0)\big) \\ &= 0 - 0 \\ p_x &= eA_x \\ &= eB_0 z \ . \end{aligned}$$

(**b**) (10 points) For simplicity assuming that the electron is nonrelativistic, and taking B_0 to be fixed, find the minimum value of the applied voltage V_0 such that the electron makes it all the way up to the top plate.

[The above describes an oversimplified version of the *static magnetron tube*, which generated the radar signals that won the Battle of Britain.]

Solution:

If the electron barely grazes the top plate, it will be travelling parallel to it, or entirely in the x direction. Since the magnetic field does no work, the electron's kinetic energy at that point will be equal to its loss of potential energy eV_0. Using the result from part (**a**),

$$\begin{aligned} eV_0 &= \frac{p_x^2}{2m} \\ V_0 &= \frac{e^2 B_0^2 d^2}{2me} \\ &= \frac{e B_0^2 d^2}{2m} \ . \end{aligned}$$

University of California, Berkeley
Physics 110B Spring 1997 (*Strovink*)

FINAL EXAMINATION

Directions. Do all seven problems, which have unequal weight. This is a closed-book closed-note exam except for two $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. Calculators are not needed. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. *You must justify what you do or say.* Express your answer in terms of the quantities specified in the problem. Box or circle your answer. Remember that when you are asked for the value of a vector quantity, you must supply both the magnitude and direction.

1. (20 points)
Write down the (real) electric and magnetic fields for a monochromatic plane wave in vacuum of amplitude E_0, angular frequency ω, and phase angle 0 (relative to a cosine). Do it for two cases: the wave is

(a) (10 points) traveling in the negative-x direction and polarized in the z direction;

(b) (10 points) traveling in the direction from the origin to the point (1,1,0), with polarization perpendicular to the x axis.

2. (10 points)
Consider a spherical pulsating bubble with constant total charge Q uniformly distributed on the surface, and with time-dependent radius $r(t) = a(1 + \epsilon \cos \omega t)$, where a, ϵ, and ω are constants. Find the total power P that is radiated.

3. (35 points)
A particle with charge e moves with speed βc around a circle of radius b centered at the origin. The circle is in the plane $z = 0$. The motion is ultrarelativistic, *i.e.* $(1 - \beta^2)^{-1/2} \gg 1$.

Liénard's equation for the Poynting vector $\mathbf{S}_\mathrm{a}$ arising from acceleration of a point particle is

$$\mathbf{S}_\mathrm{a} = (\frac{e}{4\pi\epsilon_0})^2 \frac{\epsilon_0}{c} \Big\{ \frac{\hat{\mathbf{R}}}{R^2} \big[\frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \vec{\beta}) \times \dot{\vec{\beta}}]}{(1 - \hat{\mathbf{R}} \cdot \vec{\beta})^3} \big]^2 \Big\}_\mathrm{ret} .$$

Here $\vec{\beta}c$ is the particle's velocity, $\dot{\vec{\beta}}c$ is its acceleration, $\mathbf{r}$ is a vector from the origin to the observer, $\mathbf{r}'$ is a vector from the origin to the particle, $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, and the subscript "ret" means that quantities are to be evaluated at time $t - R/c$.

(a) (20 points) Calculate the radiated power per unit area observed at $(0, 0, z)$, where $z \gg b$.

(b) (15 points) Is $\hat{\mathbf{z}}$ a direction in which the power radiated per unit solid angle is near the maximum for this motion? Explain.

4. (35 points)
Write down the Fraunhofer diffraction pattern $I(\theta)/I(\theta = 0)$ for monochromatic light of wavelength λ normally incident on a system of four thin slits. Two slits are at $y = (a \pm b)/2$, and two are at $y = -(a \pm b)/2$.

5. (35 points)
A circularly polarized plane wave of wavelength λ is normally incident on a double thin slit (separation d). In front of the top slit is placed a quarter wave plate. Obtain the Fraunhofer diffraction pattern $I(\theta)/I(\theta = 0)$. Take the optical thickness of the plate to be such that the irradiance is largest at $\theta = 0$.

6. (30 points)
Two perfect parallel mirrors enclose a sandwich consisting of two layers: a dielectric of (real constant) refractive index n between $0 < x < L$, and a region of vacuum between $L < x < (n + 1)L$. A plane standing EM wave (the sum of two traveling waves with opposite directions of propagation) propagates along the mirrors' normal. Calculate the wave's lowest possible angular frequency.

7. (35 points)
A plane wave is normally incident on an opaque screen in the plane $z = 0$. The screen blocks the semi-infinite region $x < 0$. It also has a semicircular protrusion of radius R, centered at $x = y = 0$. Thus the screen also blocks the circular region $\sqrt{x^2 + y^2} < R$.

An observer is stationed at $(0, 0, R^2/\lambda)$, where λ is the wavelength. Calculate the ratio

$$I_{\text{screen}}/I_{\text{no screen}}$$

of irradiances seen by the observer with and without the screen in place.

University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

FINAL EXAMINATION

Directions. Do all six problems, which have unequal weight. This is a closed-book closed-note exam except for three $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. Calculators are not needed. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. *You must justify what you do or say.* Express your answer in terms of the quantities specified in the problem. Box or circle your answer. Remember that when you are asked for the value of a vector quantity, you must supply both the magnitude and direction.

1. (40 points)
The total power $P(t)$ radiated by an ideal electric dipole $\mathbf{p}(t)$ is given by the Larmor formula

$$P(t) = \frac{1}{4\pi\epsilon_0}\frac{2|\ddot{\mathbf{p}}(t_{\rm ret})|^2}{3c^3} ,$$

where $t_{\rm ret}$ is the retarded time.

(a) (15 points) Consider a single positive charge e located at position $(x, y, z) = (d, 0, d\cos\omega t)$, where d and ω are constants. Approximate $d \ll \lambda$, where λ is the vacuum wavelength of the emitted radiation. Working to second order in the small quantity d/λ, compute the *time-averaged* power $\langle P\rangle$ radiated by this charge.

(b) (10 points) How much time-averaged mechanical work per unit time $\langle dW/dt\rangle$ must be exerted upon this charge in order to keep it moving as specified in **(a)**?

(c) (15 points) A second *positive* charge e is added, located at position $(-d, 0, -d\cos\omega t)$. What is the new *time-averaged* power $\langle P'\rangle$ radiated by both charges? Continue to work only to second order in the small quantity d/λ.

2. (35 points)
A plane electromagnetic wave is described by

$$\mathbf{E}(z, t) = \mathrm{Re}\Big(\tilde{\mathbf{E}}\,\exp\big(i(kz - \omega t)\big)\Big) ,$$

where

$$\tilde{\mathbf{E}} = E_0\big((2 - i)\hat{\mathbf{x}} + (1 - 2i)\hat{\mathbf{y}}\big) ,$$

and E_0, k, and ω are real constants. A linear polarizer is placed in the beam, and oriented so that the largest possible fraction of the original beam's irradiance is transmitted. What is that fraction?

3. (35 points)
A plane wave $U_0\cos(kz - \omega t)$ is incident normally on a screen. Fraunhofer conditions apply. The diffracted wave is observed from $z \to \infty$ at various angles θ with respect to the z axis.

(a) (15 points) Assume that the screen has three long parallel slits with equal spacing b and equal negligible width. Compute the irradiance ratio $I(\theta)/I(\theta = 0)$.

(b) (20 points) Instead assume that the screen has five long parallel slits with equal spacing b. The slit widths are still negligible; however, they are a function of the slit location, so that the five slit areas vary according to the ratio 1:2:3:2:1. Compute the irradiance ratio $I(\theta)/I(\theta = 0)$.

4. (20 points)
A Survivor contestant tries to signal a blimp hovering nearly overhead. It is pitch dark, and his only source of light is an infinitesimal, monochromatic, isotropic-light-emitting diode (LED). The naked LED isn't quite bright enough to be seen by his blimp-borne rescuer. Remembering Physics 110B, the contestant resolves to amplify the light signal that the rescuer perceives.

(a) (10 points) The contestant stretches a large opaque plastic sheet over a flat frame and pokes a small (couple of mm dia) circular hole in it. He carefully positions the hole directly between the LED and the blimp, separated from the LED by a couple of meters. Relative to the naked LED, is it possible that the irradiance seen by the rescuer increases? If so, by what maximum factor?

(b) (10 points) Lacking a plastic sheet, the contestant disassembles his bicycle hub to obtain a small (couple of mm dia) blackened steel ball. Using a spiderweb thread, he carefully hangs the ball directly between the LED and the blimp, separated from the LED by a couple of meters. Relative to the naked LED, is it possible that the irradiance seen by the rescuer increases? If so, by what maximum factor?

5. (35 points)
In the Drüde model for electromagnetic wave propagation in a dilute material medium, electrons (of mass m and charge $-e$) satisfy the equation of motion

$$m\ddot{x} = -\gamma m\dot{x} - kx - eE_x \ ,$$

where γ is an effective damping constant, k is an effective spring constant, and E_x is an electric field component.

Working at a particular angular frequency ω, and defining the complex electric field $\tilde{E}_x$ and complex current density $\tilde{J}_x$ through

$$\begin{aligned} E_x &\equiv \mathrm{Re}\big(\tilde{E}_x \exp(-i\omega t)\big) \\ J_x &\equiv \mathrm{Re}\big(\tilde{J}_x \exp(-i\omega t)\big) \ , \end{aligned}$$

one can then define the complex conductivity $\tilde{\sigma}$ through

$$\tilde{J}_x \equiv \tilde{\sigma}\tilde{E}_x \ .$$

In a medium having N electrons/m^3 that are so weakly bound that k is negligible, use the above information to derive the complex conductivity $\tilde{\sigma}$ as a function of angular frequency ω.
[*Hint*: Define $x \equiv \mathrm{Re}\big(\tilde{x}\exp(-i\omega t)\big)$.]

6. (35 points)
A point charge e travelling on the z axis has position

$$\begin{aligned} \mathbf{r}(t) &= +\hat{\mathbf{z}}\beta ct \ \ (t < 0) \\ &= -\hat{\mathbf{z}}\beta ct \ \ (t > 0) \ , \end{aligned}$$

where β is a positive constant that is not $\ll 1$. That is, the charge reverses direction instantaneously at $t = 0$, while it is at the origin. The fields that the charge produces are viewed by an observer at $(x, 0, 0)$, where $x > 0$.

(a) (20 points) What magnetic field $\mathbf{B}$ does the observer see at $t = 0$?

(b) (15 points) At time t such that $ct = x$ (exactly!), what is the direction of the electric field $\mathbf{E}$ seen by the observer? (You need consider only the part of the total electric field which is dominant at exactly that time.) Justify your answer.

University of California, Berkeley
Physics 110B Spring 2001 Section 1 (*Strovink*)

SOLUTION TO FINAL EXAMINATION

Directions. Do all six problems, which have unequal weight. This is a closed-book closed-note exam except for three $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. Calculators are not needed. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. *You must justify what you do or say.* Express your answer in terms of the quantities specified in the problem. Box or circle your answer. Remember that when you are asked for the value of a vector quantity, you must supply both the magnitude and direction.

1. (40 points)
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$$P(t) = \frac{1}{4\pi\epsilon_0}\frac{2|\ddot{\mathbf{p}}(t_{\rm ret})|^2}{3c^3} ,$$

where $t_{\rm ret}$ is the retarded time.

(a) (15 points) Consider a single positive charge e located at position $(x, y, z) = (d, 0, d\cos\omega t)$, where d and ω are constants. Approximate $d \ll \lambda$, where λ is the vacuum wavelength of the emitted radiation. Working to second order in the small quantity d/λ, compute the *time-averaged* power $\langle P\rangle$ radiated by this charge.
Solution:
Applying the Larmor formula to an electric dipole

$$\begin{aligned} P &= \frac{1}{4\pi\epsilon_0}\frac{2|\ddot{\mathbf{p}}(t_{\rm ret})|^2}{3c^3} \\ &= \frac{1}{4\pi\epsilon_0}\frac{2e^2d^2\omega^4\cos^2\omega t_{\rm ret}}{3c^3} \\ \langle P\rangle &= \frac{1}{4\pi\epsilon_0}\frac{e^2d^2\omega^4}{3c^3} . \end{aligned}$$

(b) (10 points) How much time-averaged mechanical work per unit time $\langle dW/dt\rangle$ must be exerted upon this charge in order to keep it moving as specified in **(a)**?
Solution:
The mechanical work done on the charge per unit time would need to supply the power that it radiates. Thus

$$\langle dW/dt\rangle = \langle P\rangle = \frac{1}{4\pi\epsilon_0}\frac{e^2d^2\omega^4}{3c^3} .$$

This answer may also be obtained by considering the radiation reaction force on the charge.

(c) (15 points) A second *positive* charge e is added, located at position $(-d, 0, -d\cos\omega t)$. What is the new *time-averaged* power $\langle P'\rangle$ radiated by both charges? Continue to work only to second order in the small quantity d/λ.
Solution:
The second charge is located on the opposite side of the origin with respect to the first charge. Thus it cancels the electric dipole moment due to the first charge. Higher-order multipole radiation may remain, but such contributions will be raised to higher powers of d/λ. Therefore, to the same order in d/λ, $\langle P'\rangle$ vanishes.

2. (35 points)
A plane electromagnetic wave is described by

$$\mathbf{E}(z,t) = \mathrm{Re}\Big(\tilde{\mathbf{E}}\,\exp\big(i(kz-\omega t)\big)\Big) ,$$

where

$$\tilde{\mathbf{E}} = E_0\big((2-i)\hat{\mathbf{x}} + (1-2i)\hat{\mathbf{y}}\big) ,$$

and E_0, k, and ω are real constants. A linear polarizer is placed in the beam, and oriented so that the largest possible fraction of the original beam's irradiance is transmitted. What is that fraction?

Solution
The beam is described by the (unnormalized) Jones vector

$$J \equiv \begin{pmatrix} 2-i \\ 1-2i \end{pmatrix} .$$

A linear polarizer with transmission axis oriented along the $\hat{\mathbf{x}}$ direction has the Jones matrix

$$M(0) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

If the polarizer's transmission axis is oriented at angle ϕ with respect to the $\hat{\mathbf{x}}$ direction, it is represented by the Jones matrix

$$\begin{aligned} M(\phi) &= R^{-1}M(0)R \\ &= \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix} , \end{aligned}$$

where the two-dimensional rotation matrix is

$$R \equiv \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} .$$

Before the polarizer, the beam irradiance I is proportional to

$$\begin{aligned} I &\propto J^\dagger J \\ &= \begin{pmatrix} 2+i & 1+2i \end{pmatrix} \begin{pmatrix} 2-i \\ 1-2i \end{pmatrix} \\ &= (4+1)+(1+4) = 10 . \end{aligned}$$

After the polarizer, the irradiance I' is proportional to

$$\begin{aligned} I' &\propto (MJ)^\dagger MJ \\ &= J^\dagger(M^\dagger M)J . \end{aligned}$$

But $M^\dagger M = M$, as can easily be verified: $M^\dagger = M$, and adding a second ideal polarizer does nothing beyond the effect of the first, so $M^2 = M$. Thus

$$\begin{aligned} I &\propto J^\dagger MJ \\ &= \begin{pmatrix} 2+i & 1+2i \end{pmatrix} \times \\ &\times \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix} \begin{pmatrix} 2-i \\ 1-2i \end{pmatrix} \\ &= 5 + 8\sin\phi\cos\phi \\ &= 9 \text{ (max)} \end{aligned}$$

when $\phi = \pi/4$. Therefore, at maximum, $I'/I = 9/10$.

3. (35 points)
A plane wave $U_0 \cos(kz - \omega t)$ is incident normally on a screen. Fraunhofer conditions apply. The diffracted wave is observed from $z \to \infty$ at various angles θ with respect to the z axis.

(a) (15 points) Assume that the screen has three long parallel slits with equal spacing b and equal negligible width. Compute the irradiance ratio $I(\theta)/I(\theta = 0)$.
Solution:
In analogy to the standard double slit problem,

$$U(\theta) \propto 1 + e^{i\beta} + e^{-i\beta} ,$$

where $\beta = kb\sin\theta$. Therefore

$$\begin{aligned} U(\theta) &\propto 1 + 2\cos\beta \\ \frac{I(\theta)}{I(0)} &= \frac{(1+2\cos\beta)^2}{9} . \end{aligned}$$

This result is equivalent to $\frac{1}{9}\sin^2(3\gamma)/\sin^2\gamma$, where $\gamma = \beta/2$.

(b) (20 points) Instead assume that the screen has five long parallel slits with equal spacing b. The slit widths are still negligible; however, they are a function of the slit location, so that the five slit areas vary according to the ratio 1:2:3:2:1. Compute the irradiance ratio $I(\theta)/I(\theta = 0)$.
Solution:
This configuration is equivalent to a triple-superposition of the triple-slit problem in (**a**), with the characteristic spacing of the superposition equal to the characteristic spacing of the slit. Therefore it is a convolution of the arrangement in (**a**) with itself. Under Fraunhofer conditions, the image is a Fourier transform of the aperture function, and the Fourier transform of a convolution is the product of the individual Fourier transforms. Therefore

$$\frac{I(\theta)}{I(0)} = \frac{(1+2\cos\beta)^4}{81} .$$

This answer may also be obtained by the brute force methods of (**a**).

4. (20 points)
A Survivor contestant tries to signal a blimp hovering nearly overhead. It is pitch dark, and his only source of light is an infinitesimal, monochromatic, isotropic-light-emitting diode (LED). The naked LED isn't quite bright enough to be seen by his blimp-borne rescuer. Remembering Physics 110B, the contestant resolves to amplify the light signal that the rescuer perceives.

(a) (10 points) The contestant stretches a large opaque plastic sheet over a flat frame and pokes a small (couple of mm dia) circular hole in it. He carefully positions the hole directly between the LED and the blimp, separated from the LED by a couple of meters. Relative to the naked LED, is it possible that the irradiance seen by the rescuer increases? If so, by what maximum factor?
Solution:
The hole could consist of an odd number of Fresnel zones (one zone would be convenient, given the rough dimensions), in which case the irradiance seen by the rescuer would be boosted by a factor of ≈ 4.

(b) (10 points) Lacking a plastic sheet, the contestant disassembles his bicycle hub to obtain a small (couple of mm dia) blackened steel ball. Using a spiderweb thread, he carefully hangs the ball directly between the LED and the blimp, separated from the LED by a couple of meters. Relative to the naked LED, is it possible that the irradiance seen by the rescuer increases? If so, by what maximum factor?
Solution:
Using the edge of the ball (as opposed to $R = 0$) as the beginning of the first Fresnel zone, and adding up the contributions of the zones, the irradiance seen by the rescuer would be approximately the same as if the ball were removed. Therefore the irradiance seen by the rescuer would not increase.
This result can also be obtained by use of Babinet's argument.

5. (35 points)
In the Drüde model for electromagnetic wave propagation in a dilute material medium, electrons (of mass m and charge $-e$) satisfy the equation of motion

$$m\ddot{x} = -\gamma m\dot{x} - kx - eE_x \ ,$$

where γ is an effective damping constant, k is an effective spring constant, and E_x is an electric field component.

Working at a particular angular frequency ω, and defining the complex electric field $\tilde{E}_x$ and complex current density $\tilde{J}_x$ through

$$\begin{aligned} E_x &\equiv \mathrm{Re}\big(\tilde{E}_x \exp(-i\omega t)\big) \\ J_x &\equiv \mathrm{Re}\big(\tilde{J}_x \exp(-i\omega t)\big) \ , \end{aligned}$$

one can then define the complex conductivity $\tilde{\sigma}$ through

$$\tilde{J}_x \equiv \tilde{\sigma}\tilde{E}_x \ .$$

In a medium having N electrons/m^3 that are so weakly bound that k is negligible, use the above information to derive the complex conductivity $\tilde{\sigma}$ as a function of angular frequency ω.
[*Hint*: Define $x \equiv \mathrm{Re}\big(\tilde{x}\exp(-i\omega t)\big)$.]
Solution:
Substituting

$$x \equiv \mathrm{Re}\big(\tilde{x}\exp(-i\omega t)\big) \ ,$$

in the equation of motion, and neglecting k with respect to $\gamma m\omega$ in view of the negligibly weak binding, we obtain

$$\begin{aligned} -m\omega^2\tilde{x} &= i\gamma m\omega\tilde{x} - e\tilde{E}_x \\ \tilde{x} &= \frac{e\tilde{E}_x/m}{\omega^2 + i\gamma\omega} \ . \end{aligned}$$

Solving for $\tilde{J}_x$,

$$\begin{aligned} J_x &= -eN\dot{x} \\ \Rightarrow \tilde{J}_x &= i\omega eN\tilde{x} \\ &= \frac{i\omega Ne^2\tilde{E}_x/m}{\omega^2 + i\gamma\omega} \\ \tilde{\sigma} &\equiv \frac{\tilde{J}_x}{\tilde{E}_x} \\ &= \frac{Ne^2/m}{\gamma - i\omega} \ . \end{aligned}$$

6. (35 points)
A point charge e travelling on the z axis has position

$$\begin{aligned}\mathbf{r}(t) &= +\hat{\mathbf{z}}\beta ct \ \ (t<0)\\ &= -\hat{\mathbf{z}}\beta ct \ \ (t>0)\ ,\end{aligned}$$

where β is a positive constant that is not $\ll 1$. That is, the charge reverses direction instantaneously at $t=0$, while it is at the origin. The fields that the charge produces are viewed by an observer at $(x,0,0)$, where $x>0$.

(a) (20 points) What magnetic field $\mathbf{B}$ does the observer see at $t=0$?
Solution:
At $t=0$, the magnetic field observed at $(x,0,0)$ was produced by the charge when it was at $t_{\rm ret}<0$, when it was still moving in the positive z direction. Therefore this is the field of a uniformly moving charge. To evaluate it, we first obtain the field in a (primed) coordinate system with its origin attached to the charge. In the primed system, the observer is located at the coordinates $(x',y',z')=(x,0,\gamma z-\gamma\beta ct)=(x,0,0)$. There the (purely electrostatic) field is given by

$$\mathbf{E}'=\hat{\mathbf{x}}\frac{e}{4\pi\epsilon_0 x^2}\ .$$

In the lab frame, using the Lorentz transformation for electromagnetic fields, the magnetic field is given by

$$\begin{aligned}B_\parallel &= B'_\parallel = 0\\ c\mathbf{B}_\perp &= \gamma c\mathbf{B}'_\perp + \gamma\vec{\beta}\times\mathbf{E}'_\perp\\ &= 0+\gamma\beta\frac{e}{4\pi\epsilon_0 x^2}\hat{\mathbf{z}}\times\hat{\mathbf{x}}\\ \mathbf{B}_\perp &= \frac{\gamma\beta}{c}\frac{e}{4\pi\epsilon_0 x^2}\hat{\mathbf{y}}\ .\end{aligned}$$

At this observation point the $\hat{\mathbf{y}}$ direction is the same as the ϕ direction, as one expects.

This answer may also be obtained by using the standard expressions for the electromagnetic field of a uniformly moving point charge, *e.g.* Griffiths 10.68-10.69.

(b) (15 points) At time t such that $ct=x$ (exactly!), what is the direction of the electric field $\mathbf{E}$ seen by the observer? (You need consider only the part of the total electric field which is dominant at exactly that time.) Justify your answer.
Solution:
At $t=x/c$, the retarded time is $t_{\rm ret}=0$. So the fields seen by the observer are the fields of a charge that is reversing the direction of its velocity (with infinite acceleration in this case). Therefore the fields at this time are dominated by the acceleration fields. For a charge accelerating along $\hat{\mathbf{z}}$, $\mathbf{E}$ is in the $\hat{\theta}$ direction, or $-\hat{\mathbf{z}}$ for this observer. However, in this problem the charge accelerates in the $-\hat{\mathbf{z}}$ direction, so $\mathbf{E}$ is along $+\hat{\mathbf{z}}$.